Stability of the collocation method applied to Volterra Integral equations with delay

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Abstract

The collocation method with polynomial splines will be applied to Volterra Integral equations with delay. Matrix transformation, formal power series and the application of Wieners Inversion theorem yield the characterization of the stability domain. This result will be compared with the results for Integral equations without delay and Differential equations with delay.

Keywords: Stability, Collocation method, Integral equations with delay.

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1 Introduction

In this paper we consider the collocation method applied to the Volterra Integral equation with constant delay $\tau > 0$

$$y(t) = \Phi(t) \text{ for } t \in [-\tau, 0)$$

and

$$y(t) = g(t) + \int_0^t k_1(t-s)\, y(s)\, ds + \int_0^{t-\tau} k_2(t-\tau-s)\, y(s)\, ds \text{ for } t \geq 0 \qquad (1)$$

with given continuous functions Φ, g, k_1 and k_2. These equations appear in various biological and physical problems.

For the collocation method the mesh is assumed to be constrained, i.e. $h = \tau/r$ for some $r \in \mathbb{N}$, which guarantees that the possible discontinuities at the multiples of

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τ of the exact solutions are at the mesh points $t_n = n \cdot h$. The collocation parameters $0 \le c_1 < \ldots < c_m \le 1$ are projected onto the subintervals $[t_n, t_{n+1}]$ and yield the collocation points $t_{n,i} = t_n + c_i h$ (hence $t_{n,i} - \tau = t_{n-r} + c_i h$). The numerical solution u has to be in the polynomial spline space, i.e. $u|_{[t_n, t_{n+1}]}$ is a polynomial of degree $m-1$. Then the approximation solution is uniquely determined with u satisfying the Volterra Integral equation at the collocation points $t_{n,i}$ and being identical with Φ on $[-\tau, 0)$. We obtain

$$u(t) = \Phi(t) \text{ for } t \in [-\tau, 0)$$

and on each subinterval $[t_n, t_{n+1}]$ an $m-$ dimensional system of implicit linear equations

$$u(t_{n,i}) = g(t_{n,i}) + \int_0^{t_{n,i}} k_1(t_{n,i} - s) u(s)\, ds + \int_0^{t_{n-r,i}} k_2(t_{n-r,i} - s) u(s)\, ds \text{ for } i = 1, \ldots, m.$$

For $t_{n,i} \in [t_r, \infty)$ the integrands $u(t)$ are piecewise polynomials, which can be written in Lagrange representation as

$$u(t_n + vh) = \sum_{l=1}^{m} u(t_{n,l}) L_l(v) \text{ for } v \in [0,1] \text{ with } L_l(v) = \prod_{i=1, i \ne l}^{m} \frac{v - c_i}{c_l - c_i} \quad .$$

For the first r intervals we have to use the known function $\Phi(t) = u(t)$ as integrand in $\int_0^{t_{n,i} - \tau}$, which we will absorb in the known vector $\vec{g}_j$.

Splitting the interval $[0, t_{n,i}]$ of integration into the subintervals $[t_j, t_{j+1}]$, $[t_n, t_{n,i}]$ and applying integral transformation to $[0,1]$ and $[0, c_i]$ leads respectively to

$$\begin{aligned} \vec{u}_n &= \vec{g}_n + h \sum_{j=0}^{n} W^{(1)}_{n-j} \vec{u}_j \qquad \text{for } n = 0, \ldots, r-1 \quad ; \\ \text{and} \quad \vec{u}_n &= \vec{g}_n + h \sum_{j=0}^{n} W^{(1)}_{n-j} \vec{u}_j + h \sum_{j=0}^{n-r} W^{(2)}_{n-r-j} \vec{u}_j \quad \text{for } n \ge r \end{aligned}$$

with

$$\begin{aligned} &\vec{u}_n = (u(t_n + c_l h))_{l=1,\ldots,m} \text{ for } n \ge 0, \\ &\vec{g}_j = (g(t_j + c_i h))_{i=1,\ldots,m} + \left(\int_0^{t_{j-r} + c_i h} k_2 \left(t_{j-r,i} - s \right) \Phi(s)\, ds \right)_{i=1,\ldots,m} \quad \text{for } j \le r-1, \\ &\vec{g}_j = (g(t_j + c_i h))_{i=1,\ldots,m} \qquad \text{for } j \ge r, \\ &W_0^{(s)} = \left(\int_0^{c_i} k_s((c_i - v)h) L_l(v)\, dv \right)_{\substack{i=1,\ldots,m \\ l=1,\ldots,m}} \qquad s = 1,2\,, \\ &W_j^{(s)} = \left(\int_0^{1} k_s(t_{j,i} - vh) L_l(v)\, dv \right)_{\substack{i=1,\ldots,m \\ l=1,\ldots,m}} \qquad s = 1,2 \quad \text{for } j \ge 1 \quad . \end{aligned}$$

If we rewrite this in formal power series (using the notation $v(\xi) = \sum_{n=0}^{\infty} v_n \xi^n$) we obtain

$$\vec{u}(\xi) = \vec{g}(\xi) + h \left(W^1(\xi) + \xi^r W^2(\xi) \right) \vec{u}(\xi) \quad . \tag{2}$$

Essentially herefore was the linear convolution kernels in the Volterra Integral equation.

2 Stability notation

While the convergence of the introduced collocation method is analyzed by Brunner [1994] in details, we will discuss its stability behavior. Note that the known stability analysis for numerical methods deal with Integral equations which can be reduced to differential equations.

We will consider the non reducible test equation (analogously to the differential equations)

$$y(t) = \Phi(t) \text{ for } t \in [-\tau, 0)$$

and

$$y(t) = g(t) + a\int_0^t y(s)\,ds + b\int_0^{t-\tau} y(s)\,ds \text{ for } t \geq 0 \quad . \tag{3}$$

Since we consider constant kernels a and b the weights of the collocation method simplify. Define

$$W := \left(\int_0^{c_i} L_l(v)\,dv\right)_{\substack{i=1,\dots,m \\ l=1,\dots,m}} \quad \text{and } \mathrm{L} := \left(\int_0^{1} L_l(v)\,dv\right)_{\substack{i=1,\dots,m \\ l=1,\dots,m}}$$

we obtain

$$D(\xi) := W^1(\xi) + \xi^r W^2(\xi) = (a + b\xi^r)\left(W + L\frac{\xi}{1-\xi}\right) \quad .$$

The collocation equation 2 can be written as

$$\vec{u}(\xi) = \vec{g}(\xi) + h(a + b\xi^r)\left(W + L\frac{\xi}{1-\xi}\right)\vec{u}(\xi) = \vec{g}(\xi) + hD(\xi)\vec{u}(\xi) \quad . \tag{4}$$

The following definition preserves with the independence from r, that the asymptotic stability behavior of the exact solution y for 3 is independent from τ.

Definition 1 *The stability domain S of the collocation method applied to the Integral equation 3 consists of all $(\alpha, \beta) \in \mathbb{C}^2$, $\alpha = ah$, $\beta = bh$ with any $r \in \mathbb{N}$, for which* $\lim_{t\to\infty} u(t) = 0$ *whenever* $\lim_{t\to\infty} g(t) = c$.

3 Matrix transformation and Characterization of the stability domain

The difficulties to analyze the stability domain of the collocation method mainly result out of two facts. Firstly the method for the Integral equation is not reducible to a finite recurrence relation as it is the case for Differential equations, (i.e. they do not yield a summation of fixed length k). Hence the usual approach with the characteristic polynomial is not possible. Secondly $D(\xi)$ has coefficient sequences, which are not in l^1, (i.e. $D(\xi)$ is not absolute convergent for $|\xi| = 1$). This is caused by the kernels a and b, which are not in $L^1(\mathbb{R}^+)$.

To reduce the divergence at $|\xi| = 1$ to only one coefficient in the matrix $D(\xi)$, we apply the following matrix transformation $BD(\xi)B^{-1}$ with ($\mathbb{L}_i = \int_0^1 L_i(v)dv$)

$$B = \begin{pmatrix} 1 & & & & & -1 \\ & . & & 0 & & . \\ & & . & & & . \\ & 0 & & . & & . \\ & & & & 1 & -1 \\ \mathbb{L}_1 & . & . & . & . & \mathbb{L}_m \end{pmatrix} \Rightarrow B^{-1} = \begin{pmatrix} 1-\mathbb{L}_1 & -\mathbb{L}_2 & . & . & -\mathbb{L}_{m-1} & 1 \\ -\mathbb{L}_1 & 1-\mathbb{L}_2 & . & & . & 1 \\ . & -\mathbb{L}_2 & . & . & . & . \\ . & & . & . & -\mathbb{L}_{m-1} & . \\ . & & . & . & 1-\mathbb{L}_{m-1} & . \\ -\mathbb{L}_1 & . & . & . & -\mathbb{L}_{m-1} & 1 \end{pmatrix}$$

Remark 1 *This matrix transformation corresponds to the change of the basis polynomials* $\{L_1(v), \ldots, L_m(v)\}$ *to the polynomials* $\{L_1(v) - \mathbb{L}_1, \ldots, L_{m-1}(v) - \mathbb{L}_{m-1}, 1\}$ *for the representation of* $u(t)$.

Define $\vec{v}_n = B \cdot \vec{u}_n$ then

$$u(t_n + vh) = \vec{u}_n^t \cdot \begin{pmatrix} L_1(v) \\ . \\ . \\ L_m(v) \end{pmatrix} = \vec{v}_n^t \begin{pmatrix} L_1(v) - \mathbb{L}_1 \\ . \\ . \\ L_{m-1}(v) - \mathbb{L}_{m-1} \\ 1 \end{pmatrix}$$

since $\sum_{l=1}^m L_l(v) = 1$. The collocation equation yield

$$\vec{v}(\xi) = B\vec{g}(\xi) + hBD(\xi)B^{-1}\vec{v}(\xi)$$

with

$$BD(\xi)B^{-1} = (a + b\xi^r)\left(A + E\frac{\xi}{1-\xi}\right) \quad ;$$

$$A := BWB^{-1} = \begin{pmatrix} \underset{\substack{i=1,\ldots,m-1 \\ s=1,\ldots,m-1}}{-\int_{c_s}^{c_m} L_i(v)\,dv + (c_m - c_s)\,\mathbb{L}_i} & \underset{s=1,\ldots,m-1}{c_s - c_m} \\ & \\ \sum_{l=1}^m \mathbb{L}_l \int_0^{c_l} L_i(v)\,dv - \frac{1}{2}\mathbb{L}_i & \frac{1}{2} \end{pmatrix}$$

and

$$E := \begin{pmatrix} & & & 0 \\ & 0 & & . \\ & & & 0 \\ 0 & . & 0 & 1 \end{pmatrix} \quad .$$

The advantage of this transformation is therefore also, that the summation of the products of matrices with vectors is reduced to the summation of just one coefficient.

It is obvious that $\lim_{t\to\infty} u(t) = 0$ if and only if $\lim_{n\to\infty} ||\vec{v}_n||_\infty = 0$, i.e. $\vec{v}(\xi)$ has coefficient sequences in c_0.

We have

$$\left(I - hBD(\xi)B^{-1}\right)\vec{v}(\xi) = B\vec{g}(\xi) \quad .$$

To apply Wieners Inversion Theorem (Zygmund [1959]):

if $f(t) \neq 0$ and the Fourier series of f is absolute convergent, then the Fourier series of $1/f$ is absolute convergent.

Let define

$$C(\xi) = \begin{pmatrix} 1 & & & \\ & \cdot & & 0 \\ & & 1 & \\ 0 & & & (1-\xi) \end{pmatrix}$$

and consider

$$C(\xi)\left[I - hBD(\xi)B^{-1}\right]\vec{v}(\xi) = C(\xi)B\vec{g}(\xi) \quad .$$

We obtain

$$C(\xi)\left[I - hBD(\xi)B^{-1}\right] = C(\xi) - h\left(a + b\xi^r\right)(A - \xi EA + \xi E)$$

has coefficient sequences in l^1 and $C(\xi)B\vec{g}(\xi)$ has coefficient sequences in c_0 since $g(t_{n,i}) - g(t_{n,m}) \to 0$ as $n \to \infty$ and

$$\begin{aligned} &(1-\xi)\left(\sum_{n=0}^{\infty}\left(\sum_{i=1}^{m} \mathbb{L}_i g(t_{n,i})\right)\xi^n\right) \\ &= (1-\xi)\left(\sum_{n=0}^{\infty}\left(\sum_{i=1}^{m} \mathbb{L}_i g(t_{n,i}) - c\right)\xi^n\right) + (1-\xi)c\frac{1}{1-\xi} = c + (1-\xi)\sum_{n=0}^{\infty} s_n\xi^n \end{aligned}$$

with $s_n \to 0$ as $n \to \infty$. Wieners Inversion Theorem and the fact $l^1 * c_0 \subset c_0$ applied to

$$\vec{v}(\xi) = \left[C(\xi)\left(I - hBD(\xi)B^{-1}\right)\right]^{-1}\left[C(\xi)B\vec{g}(\xi)\right]$$

yields

Theorem 1 *If for all $|\,\xi\,| \leq 1$ we have*

$$det\left(C(\xi)\left(I - hBD(\xi)B^{-1}\right)\right) \neq 0$$

then $\lim_{t\to\infty} u(t) = 0$ *whenever* $\lim_{t\to\infty} g(t) = c$.

We can eliminate $C(\xi)$ in the sufficient condition.

Theorem 2 *The following inequalities are equivalent:*

$$det\left(C(\xi)\left(I - hBD(\xi)B^{-1}\right)\right) \neq 0 \Leftrightarrow det\left(I - hBD(\xi)B^{-1}\right) \neq 0 \quad .$$

Proof: For $\xi \neq 1$ it is obvious, since all determinants exist. For $\xi = 1$ expand $\det(C(1) - h(a+b)(A - EA + E))$ in respect to the last row. On the other hand expand $\det(I - hBD(\xi)B^{-1})$ in respect to the last column and take the limit as $\xi \to 1$. In both cases we get

$$h(a+b)\det(I_{m-1} - h(a+b)A_{m,m}) \neq 0$$

where $A_{m,m}$ is the matrix A, where the last row and column is dropped. Hence the inequalities are equivalent.

□

Furthermore, the necessity of the above condition for stability can be shown by adopting the ideas of theorem 3.4 in Blank [1993a]. We omit the details for the sake of shortness. This yields

Theorem 3 (Characterization of the stability domain) *The stability domain is characterized by*

$$S = \Big\{(\alpha, \beta) \in \mathbb{C}^2 \quad | \quad \alpha = ah, \beta = bh \text{ with} \\ det\left(I - hBD(\xi)B^{-1}\right) \neq 0 \text{ for all } \ | \xi | \leq 1; r \in \mathbb{N}\Big\} \quad .$$

4 Comparison with non delay Integral equation

The above characterization of the stability domain is given by the zeros of a polynomial with complex power series as coefficient, which depends on a, b, r and ξ. To analyze this characterization we can compare it with the non delay Volterra Integral equation (i.e. $b = 0$),

$$y(t) = g(t) + a \int_0^t y(s)\, ds \quad \text{and} \quad y(0) = y_0 \quad . \tag{5}$$

We obtain the following condition for its stability domain S_{ND}

$$ha \in S_{ND} \Leftrightarrow \det\left(I - ha\left(A + E\frac{\xi}{1-\xi}\right)\right) \neq 0 \qquad \text{for all } \ | \xi | \leq 1 \quad .$$

For the delay equation we have to substitute ha with $h(a + b\xi^r)$. Since $\xi \mapsto h(a + b\xi^r)$ maps $\overline{B}_1(0)$ into $\overline{B}_{|hb|}(ha)$ for all $r \in \mathbb{N}$ we have

Corollary 1 *If $\overline{B}_{|\beta|}(\alpha) \subseteq S_{ND}$ then $(\alpha, \beta) \in S$.*

The theorem below (see Blank [1993b]) gives the basis to plot the stability domain for the collocation method applied to the non delay Integral equation.

Theorem 4

$$\partial S_{ND} \subseteq \{ha \in \mathbb{C} \mid det\left(I - BD(\xi)B^{-1}\right) = 0 \textit{ for some } \ | \xi | = 1\} \quad .$$

Hence, for the Volterra Integral equation with delay we can graph a domain which is in the stability domain S by plotting to each $\mid \beta \mid$ the corresponding curve to ∂S_{ND} with distance $\mid \beta \mid$. We obtain graphs with the axis $Re(\alpha), Im(\alpha)$ and $\mid \beta \mid$ (see sketch on the left). Or we can plot the corresponding level sets to $\mid \beta \mid$ (see sketch on the right).

Furthermore we obtain

Corollary 2 *If the collocation method is $A-$stable for the non delay Integral equation 5 , i.e. $\mathbb{C}^- \subseteq S_{ND}$, then the method is $A-$stable for the delay Integral equation 3 i.e. $\{(\alpha,\beta) \in \mathbb{C}^2 \mid\mid \beta \mid < -Re(\alpha)\} \subseteq S$.*

5 Comparison with Delay Differential Equations

The essential difference between the Integral equation and the corresponding Differential equation with delay

$$y(t) = \Phi(t) \text{ for } t \in [-\tau, 0)$$

and

$$y'(t) = ay(t) + by(t-\tau) \text{ for } t \geq 0 \tag{6}$$

is the function $g(t)$. For the rewritten Differential equation we have $g(t) \equiv c$, while for the stability analysis to the more general Integral equation 3 we consider $\lim_{t\to\infty} g(t) = c$. Hence, the stability domain S_{DE} for the collocation method applied to the Differential Equation with delay includes the stability domain S. However it can be shown

Theorem 5 *The stability domains for the collocation method to the Integral equation 3 with constant kernels and to the Differential equation 6 are identical ($S_{DE} = S$).*

Proof: In Blank [1991] it is shown

$$\det\left(I - z\left(A + E\frac{\xi}{1-\xi}\right)\right) \neq 0$$

$$\Leftrightarrow \frac{1}{\xi} \neq R(z) = 1 + z\,(\mathbb{L}_1, \ldots, \mathbb{L}_m)\,(I - zW)^{-1}\,(1, \ldots, 1)^t$$

Setting $z = ha + hb\xi^r$ we obtain the inequality, which characterizes the stability domain of the Differential equation with delay (see Zennaro [1986]). $\square$

6 Remarks

We have the following relations between the stability domains:

$$\begin{array}{lcl} \text{Volterra Integral equations with delay} & \longleftarrow & \text{Volterra Integral equations without delay} \\ \updownarrow & & \updownarrow \\ \text{Differential Equation with delay} & \longleftarrow & \text{Differential Equation without delay} \end{array}$$

The known results for Differential equations consider special choices of collocation parameters and analyze the $A-$ or $A(\theta)-$stability while theorem 4 for the Integral equation without delay enables to plot numerically the stability domain for any choice of collocation parameter.

Furthermore, the presented way of analyzing the stability domain can be extended to non-constant convolution kernels. If the considered kernel is in $L^1(\mathbb{R}^+)$ there will appear no problem. If the kernel is not in $L^1(\mathbb{R}^+)$ the difficulty will be to find the auxiliary matrix $C(\xi)$. For example for the weakly singular kernel $t^{-\alpha}$ with $\alpha \in [0, 1)$ we would choose the same matrix $C(\xi)$ with the substitution $(1 - \xi)$ by $(1 - \xi)^{1-\alpha}$. The results would be respectively the same.

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