Stability of collocation for weakly

singular Volterra equations

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Abstract. For Abel integral equations of the second kind, we investigate the stability of the collocation method with polynomial splines. We characterize the stability domain in terms of the eigenvalues of the power series, which arises from the employed method. Futhermore, we prove strong stability and provide a rigorous discussion of the boundary of the stability region. A few examples are considered in more depth and numerical illustrations are given.

Subject Classifications. AMS: 65R20, 45D05, 45L10.

1. Introduction

1.1 Summary of results: We examine the stability of collocation methods based on piecewise polynomials applied to the familiar Volterra–Abel equation of the second kind (e.g. Brunner and van der Houwen [1986])

$$y(t) \;=\; g(t) + \lambda \cdot \int_0^t (t-s)^{-\alpha} \cdot y(s)\; ds \quad (t \geq 0) \tag{1.1}$$

where $\alpha \in [0,1)$ and g is a given function. The numerical method yields a system of discrete equations of the form

$$\vec{u}_n = \vec{g}_n + \lambda h^{1-\alpha} \sum_{j=0}^{n} W_{n-j} \vec{u}_j \quad (n \geq 0)$$

in which $W_n \in \mathbb{R}^{m \times m}$ (section 2). We show that the numerical method is stable (in a sense defined) if and only if

$$\det \left(I - \lambda h^{1-\alpha} W(\xi) \right) \neq 0 \quad \text{ for all } \; |\xi| \leq 1$$

where $W(\xi) := \sum_{n=0}^{\infty} W_n \xi^n$. Similar results occur in the scalar case ($m = 1$) for methods based on Linear Multistep Methods (Lubich [1983, 1986]), in the case of positive definite $L^1(\mathbb{R}_+)$–kernels for Runge–Kutta methods (Hairer and Lubich [1984]) and, more widely, in the case $W(\xi)$ is a polynomial in ξ. We show how to overcome technical difficulties arising from the fact that in our case $W(\xi)$ is not convergent for $\xi = 1$ (section 3).

We prove that the method is strongly stable and also deduce a characterization for the boundary of the stability domain. The first property ensures that for h sufficiently small we obtain stability (section 4). Results on the boundary of stability regions in simpler situations can be found in the literature. In our work we provide a rigorous discussion of the boundary of the stability region, that overcomes gaps in the existing published literature. To this end, we employ functions defined on certain compact Riemann surfaces (section 5). (As an incidental comment, we note that for $\alpha = 0$ the results reduce to those for an equivalent implicit Runge–Kutta method for $y'(t) = \lambda y(t)$ see Blank [1991]). The theoretical discussion is illustrated by numerical examples (section 6).

The paper is based on the thesis of the author and further details can be found in Blank [1991].

1.2 The analytic solution: The following theorem is due to Lubich [1986].

Theorem 1.1. *Let $g \in C([0,\infty))$. Then there exists a unique continuous solution y to (1.1). Furthermore, if the parameter λ satisfies the condition*

$$|\arg(\lambda) - \pi| < \frac{1}{2}(1+\alpha)\pi$$

then y is stable (i.e. y tends to zero as $t \to \infty$), when g possesses a finite limit for $t \to \infty$ and y is bounded, when g is bounded.

Note, the kernel $t^{-\alpha}$ is not continuous at the origin, thereby leading, in general, to nonsmooth solutions at $t = 0$; it is not in $L^1(\mathbb{R}^+)$ and this has strong implications for the stability of the analytic solution, which are reflected in our discrete theory.

So far, most stability results of methods for integral test equations consider equations which reduce to ordinary differential equations or methods which have finite recursion relations. In contrast to this, we consider the collocation method which leads to an infinite recursion relation.

2. Collocation approximation and notions of stability

Assume Z to be a uniform partition of $\mathbb{R}_+$, and denote by Π_m the set of polynomials of degree at most $m-1$. Let the collocation parameters be given by $0 \leq c_1 < \cdots < c_m \leq 1$. Their projection on $[t_n, t_{n+1}]$ determines the collocation points $t_n + c_i h$ for $i = 1, .., m$. The polynomial spline collocation applied to the Volterra integral equation (1.1) yield a function u for the approximation of the solution y of (1.1), which is in the space of the polynomial splines $S_m(Z) = \left\{u : \ u|_{[t_n,t_{n+1})} \in \Pi_m\right\}$ and fulfills the equation (1.1) in the discrete set of the collocation points. More precisely, the m free coefficients of u in each subinterval $[t_n, t_{n+1})$ are determined by requiring

$$u(t_n + c_i h) = g(t_n + c_i h) + \lambda \int_0^{t_n + c_i h} (t_n + c_i h - s)^{-\alpha}\, u(s)\, ds \quad (i = 1, \cdots, m)\ .$$

(For further discussion see Brunner and van der Houwen [1986]). Throughout this paper we define:

the Lagrange polynomial $L_{c_l}(v) := \prod_{\substack{k \neq l \\ k=1}}^{m} (v - c_k)/(c_l - c_k)$,

the weights $$w_0(c_i, c_l) := \int_0^{c_i} (c_i - v)^{-\alpha} L_{c_l}(v)\, dv$$

and $$w_j(c_i, c_l) := \int_0^{1} (j + c_i - v)^{-\alpha} L_{c_l}(v)\, dv \quad \text{for } j \geq 1 .$$

With the Lagrange representation of u in each subinterval and with (the superscript ^{t} denotes transpose)

$$\vec{u}_n = (u(t_n + c_1 h), \cdots, u(t_n + c_m h))^t \in \mathbb{C}^m , \quad \vec{g}_n = (g(t_n + c_1 h), \cdots, g(t_n + c_m h))^t \in \mathbb{C}^m$$

$$\text{and} \quad W_j = \Big(w_j(c_i, c_l) \Big)_{\substack{i=1,\cdots,m \\ l=1,\cdots,m}} \in \mathbb{R}^{m \times m}$$

we obtain for $n \geq 0$ the infinite recursions relation

$$\vec{u}_n = \vec{g}_n + \lambda h^{1-\alpha} \sum_{j=0}^{n} W_{n-j} \vec{u}_j \quad . \tag{2.1}$$

The convolution of the sequence $(W_j)_{j \in \mathbb{N}_0}$ with the sequence $(\vec{u}_j)_{j \in \mathbb{N}_0}$ is essential to the analysis of stability. Therefore, denoting the formal power series by $\vec{u}(\xi) := \sum_{n=0}^{\infty} \vec{u}_n \xi^n$, $\vec{g}(\xi) := \sum_{n=0}^{\infty} \vec{g}_n \xi^n$ and $W(\xi) := \sum_{n=0}^{\infty} W_n \xi^n$, the sequence $(\vec{u}_j)_{j \in \mathbb{N}_0}$ is given by

$$\vec{u}(\xi) = \vec{g}(\xi) + \lambda h^{1-\alpha} W(\xi) \cdot \vec{u}(\xi) \quad . \tag{2.2}$$

We consider these power series to obtain properties for the sequences of coefficients.

Stability concepts relate to the response of solutions to "admissible" pertubations. For linear equations, this is equivalent to considering the qualitative behaviour of solutions where the function g has certain behaviour. In consequence, we can adopt the following definitions:

Definition 2.1.

a.) The stability domain S of the collocation method (2.1) of the Volterra integral equation (1.1) consists of all $\lambda h^{1-\alpha} \in \mathbb{C}$, for which the numerical solution $u(t)$ converges to 0 as $t \to \infty$, whenever $g(t) \to$ const. as $t \to \infty$.

b.) The method is said to be A-stable if

$$\left\{ \lambda h^{1-\alpha} \in \mathbb{C} \mid h > 0 \, , \; |\arg(\lambda) - \pi| < \frac{1}{2}(1+\alpha)\pi \right\} \subseteq S \quad .$$

c.) The method is said to be $A(\theta)$-stable if S includes the region $|\arg(z) - \pi| < \theta$.

d.) The method is said to be strongly stable, if for all $\lambda \in \mathbb{C}$ with $|\arg(\lambda) - \pi| < \frac{1}{2}(1+\alpha)\pi$ there exists a $h_0(\lambda) > 0$ with $\lambda h^{1-\alpha} \in S$ for all $0 < h \le h_0(\lambda)$.

3. Characterization of the stability domain

We shall use the notation l^1, l^∞ and c_0 to denote the linear space of sequences, where l^1 and l^∞ are defined as usual and c_0 is the space of sequences converging to zero. Furthermore, we shall need to appeal to the following result (Lubich [1986]).

Lemma 3.1. *Let $(a_n)_{n \in \mathbb{N}} \in l^1$ and $|\xi_0| \le 1$ then the sequence of coefficients of $b(\xi) = (a(\xi) - a(\xi_0)) \, / \, (\xi - \xi_0)$ are in c_0.*

According to (2.2), we have $(I - \lambda h^{1-\alpha} W(\xi))\vec{u}(\xi) = \vec{g}(\xi)$. The coefficients of $W(\xi)$ possess a radius of convergence unity but in particular for $\xi = 1$ the series is not absolutely convergent. Therefore, to apply the Wiener inversion theorem (Zygmund [1959], p.245) for absolutely convergent Fourier series to the collocation equation we introduce

$$C(\xi) = \begin{pmatrix} 1 & & & & -1 \\ & . & 0 & & . \\ & 0 & . & & . \\ & & & 1 & -1 \\ 0 & . & . & 0 & (1-\xi)^{1-\alpha} \end{pmatrix}$$

and consider, in the following, the equation

$$C(\xi) \left(I - \lambda h^{1-\alpha} W(\xi) \right) \vec{u}(\xi) = C(\xi) \, \vec{g}(\xi) \quad .$$

Note that $C(\xi)$ has coefficient sequences in l^1 since the coefficient sequence of $(1-\xi)^{1-\alpha}$ is in l^1 (Erdélyi [1953]). We have the following sufficient condition for stability.

Theorem 3.2. *If* $\det\left(C(\xi)\left(I-\lambda h^{1-\alpha}W(\xi)\right)\right)\neq 0$ *for all* $|\xi|\leq 1$ *then* $\lambda h^{1-\alpha}$ *is in the stability domain* S*.*

Proof. Since $w_j(c_i,c_l)=j^{-\alpha}\int\limits_0^1 L_{c_l}(v)\;dv+e_j(c_i,c_l)$ for $j\geq 1$ with $(e_j(c_i,c_l))_{j\in\mathbb{N}}\in l^1$ we observe that

$$C(\xi)\,W(\xi)=\begin{pmatrix}[w(c_i,c_l)(\xi)-w(c_m,c_l)(\xi)]_{\substack{i=1,\cdots,m-1\\ l=1,\cdots,m}}\\ \\ [(1-\xi)^{1-\alpha}w(c_m,c_l)(\xi)]_{l=1,\cdots,m}\end{pmatrix}$$

has coefficient sequences in l^1 (Erdélyi [1953], p.47). Therefore the coefficient sequence of $\det\left(C(\xi)\left(I-\lambda h^{1-\alpha}W(\xi)\right)\right)$ is in l^1. Provided that $\det\left(C(\xi)\left(I-\lambda h^{1-\alpha}W(\xi)\right)\right)\neq 0$ for all $|\xi|\leq 1$, we can apply the Wiener Inversions theorem to this series. Consequently $\frac{1}{\det(C(\xi)(I-\lambda h^{1-\alpha}W(\xi)))}$ is for all $|\xi|\leq 1$ absolutely convergent and the coefficient sequence is in l^1. This yields

$$\left[C(\xi)\left(I-\lambda h^{1-\alpha}W(\xi)\right)\right]^{-1}=\frac{1}{\det\left(C(\xi)\left(I-\lambda h^{1-\alpha}W(\xi)\right)\right)}\cdot\operatorname{adj}\;\left(C(\xi)\left(I-\lambda h^{1-\alpha}W(\xi)\right)\right)$$

has coefficient sequences in l^1. Furthermore we have

$$C(\xi)\,\vec{g}(\xi)=\begin{pmatrix}[\sum\limits_{j=0}^{\infty}(g(t_j+c_ih)-g(t_j+c_mh))\xi^j]_{i=1,\cdots,m-1}\\ \\ (1-\xi)^{1-\alpha}\sum\limits_{j=0}^{\infty}g(t_j+c_mh)\xi^j\end{pmatrix}\quad .$$

We know $(g(t_j+c_ih)-g(t_j+c_mh))_{j\in\mathbb{N}_0}\in c_0$ and

$$\begin{aligned}(1-\xi)^{1-\alpha}\sum_{j=0}^{\infty}g(t_j+c_mh)\xi^j&=(1-\xi)^{1-\alpha}(1-\xi)^{-1}\lim_{k\to\infty}g(t_k+c_mh)\\ &\quad+(1-\xi)^{1-\alpha}\sum_{j=0}^{\infty}\lim_{k\to\infty}\left(g(t_j+c_mh)-g(t_k+c_mh)\right)\xi^j\end{aligned}$$

has a coefficient sequence in c_0 because $(1-\xi)^{-\alpha}$ has a coefficient sequence in c_0, $\left(\lim\limits_{k\to\infty}g(t_j+c_mh)-g(t_k+c_mh)\right)_{j\in\mathbb{N}_0}\in c_0$ and $l^1*c_0\subseteq c_0$. Hence $C(\xi)\,\vec{g}(\xi)$ has coefficient sequences in c_0. We can conclude that the coefficient sequences of $\vec{u}(\xi) = \left[C(\xi)\left(I - \lambda h^{1-\alpha} W(\xi)\right)\right]^{-1} \cdot \left[C(\xi)\,\vec{g}(\xi)\right]$ are in c_0. Hence $u(t)$ converges to 0 and $\lambda h^{1-\alpha} \in S$. ∎

If we consider the condition $\det\left(C(\xi)\left(I - \lambda h^{1-\alpha} W(\xi)\right)\right) \neq 0$ for all $|\xi| \leq 1$, we have to pay attention to the product of the power series, not to the determinants of each matrix. (There is no difference if $\xi \neq 1$.) If $\xi = 1$, $C(\xi)$ is a singular matrix and $W(\xi)$ does not exist. However, $C(\xi)\,W(\xi)$ exists and is not zero. In the case $\xi \neq 1$, the condition of theorem 3.2 reduces to the consideration of the eigenvalues of $W(\xi)$. In the case $\xi = 1$, the questions arise: How do the eigenvalues behave? Are all m eigenvalues unbounded? With the following matrix transformation $BW(\xi)\,B^{-1}$, we reduce the problem of the unbounded series, which are defined as coefficients of the matrix $W(\xi)$, to one coefficient, which is an unbounded series. Hence the matrix transfomation is a fundamental basis for all further analysis of the eigenvalues and, accordingly, of the stability domain. The factors influencing the choice of the matrix transformation are that the singularity of $w(c_i, c_l)(\xi)$ depends only on c_l and that the Lagrange polynomials add up to 1. We set $\mathbb{L}_{c_l} = \int\limits_0^1 L_{c_l}(v)\ dv$ and define

$$B := \begin{pmatrix} 1 & & & & & -1 \\ & . & & 0 & & . \\ & & . & & & . \\ & 0 & & . & & . \\ & & & & 1 & -1 \\ \mathbb{L}_{c_1} & . & . & . & . & \mathbb{L}_{c_m} \end{pmatrix} \in \mathbb{R}^{m \times m}$$

$$\text{with inverse} \quad B^{-1} = \begin{pmatrix} 1 - \mathbb{L}_{c_1} & -\mathbb{L}_{c_2} & . & . & -\mathbb{L}_{c_{m-1}} & 1 \\ -\mathbb{L}_{c_1} & 1 - \mathbb{L}_{c_2} & . & & . & 1 \\ . & -\mathbb{L}_{c_2} & . & . & . & . \\ . & & . & . & -\mathbb{L}_{c_{m-1}} & . \\ . & & . & . & 1 - \mathbb{L}_{c_{m-1}} & . \\ -\mathbb{L}_{c_1} & . & . & . & -\mathbb{L}_{c_{m-1}} & 1 \end{pmatrix} .$$

Hereby we can established the following result (for details see Blank [1991]):

Theorem 3.3.

$$BW(\xi)\,B^{-1} = \begin{pmatrix} A(\xi) & \vec{a}_1(\xi) \\ & \\ \vec{a_2}^{\,t}(\xi) & a_{mm}(\xi) \end{pmatrix} ,$$

where $A(\xi) = \left(a_{is}(\xi)\right)_{\substack{i=1,\cdots,m-1\\ s=1,\cdots,m-1}}$, $\vec{a_1}(\xi) = (a_{im}(\xi))_{i=1,\cdots,m-1}$, $\vec{a_2}(\xi) = (a_{ms}(\xi))_{s=1,\cdots,m-1}$

have coefficient sequences in l^1

and $a_{mm}(\xi) = \sum_{k=1}^{m} I\!L_{c_k} \sum_{l=1}^{m} w(c_k, c_l)(\xi) = (1-\xi)^{\alpha-1}\Gamma(1-\alpha) + v(\xi)$ *where* $v(\xi)$ *has a coefficient sequence in* l^1*. Furthermore, for* $\alpha \neq 0$ *we have* $\vec{a}_1(1) = \vec{0} \in \mathbb{C}^{m-1}$.

We are now in a position to derive that the sufficient condition for stability depends only on $W(\xi)$.

Lemma 3.4.

$$\det\left(C(\xi)\left(I - \lambda h^{1-\alpha} W(\xi)\right)\right) \neq 0 \quad \Leftrightarrow \quad \frac{1}{\lambda h^{1-\alpha}} \textit{ is not an eigenvalue of } W(\xi) \quad \textit{for all } |\xi| \leq 1 \, .$$

For $\xi = 1$ *we denote by the eigenvalues of* $W(\xi)$ *the limits of the eigenvalues as* $\xi \to 1$.

Proof.

If $\xi \neq 1$ the assertion is an immediate consequence of $\det C(\xi) \neq 0$.

If $\xi = 1$ we consider:

$$\lim_{\xi \to 1} \det\left(C(\xi)\left(I - \lambda h^{1-\alpha} W(\xi)\right)\right)$$

$$= \lim_{\xi \to 1} \det\left[\begin{pmatrix} 1 & & & \\ & \cdot & 0 & \\ & & \cdot & \\ & 0 & 1 & \\ & & & (1-\xi)^{1-\alpha} \end{pmatrix}\left(I - \lambda h^{1-\alpha} B W(\xi) B^{-1}\right)\right]$$

$$= \lim_{\xi \to 1} \det\begin{pmatrix} I_{m-1} - \lambda h^{1-\alpha} A(\xi) & -\lambda h^{1-\alpha} \vec{a}_1(\xi) \\ -\lambda h^{1-\alpha}(1-\xi)^{1-\alpha} \vec{a_2}^{\,t}(\xi) & (1-\xi)^{1-\alpha} - \lambda h^{1-\alpha}(1-\xi)^{1-\alpha} a_{mm}(\xi) \end{pmatrix}$$

$$= -\lambda h^{1-\alpha} \lim_{\xi \to 1}\left((1-\xi)^{1-\alpha} a_{mm}(\xi)\right) \cdot \det\left(I_{m-1} - \lambda h^{1-\alpha} A(1)\right) \quad ,$$

since $\vec{a}_2(\xi)$ has coefficient sequences in l^1. This implies

$$\lim_{\xi \to 1} \det\left(C(\xi)\left(I - \lambda h^{1-\alpha} W(\xi)\right)\right) \neq 0$$

$$\Leftrightarrow \quad \frac{1}{\lambda h^{1-\alpha}} \text{ is not an eigenvalue of } A(1) \quad \text{and} \quad \lambda h^{1-\alpha} \neq 0 \quad .$$

We now turn our attention to the equivalence of the previous formula to

$$\frac{1}{\lambda h^{1-\alpha}} \text{ is not an eigenvalue of } BW(1)B^{-1} \quad .$$

If $\alpha \neq 0$ this follows from $\vec{a}_1(1) = 0$ and $\lim_{\xi \to 1} a_{mm}(\xi) = \infty$. If $\alpha = 0$ we find that all coefficients of $BW(\xi)B^{-1}$ are constant except $a_{mm}(\xi)$. Thus we obtain with $\frac{1}{a_{mm}(\xi)} \to 0$ for $\xi \to 1$ the equivalence to $0 \neq \lambda h \cdot \det(I_{m-1} - \lambda h A(1))$. ∎

Finally, with the following proof of the necessity of the condition $\det(I - zW(\xi)) \neq 0$ for all $|\xi| \leq 1$ for stability ($z \in S$) we obtain the characterization of the stability domain.

Theorem 3.5. *The stability domain S of the collocation method with polynomial splines on a uniform partition applied to the Volterra integral equation (1.1) is given by*

$$S = \mathbb{C} \setminus \left\{ \lambda h^{1-\alpha} \mid \frac{1}{\lambda h^{1-\alpha}} \textit{ is an eigenvalue of } W(\xi) \textit{ for a } \xi \in \mathbb{C} \textit{ with } |\xi| \leq 1 \right\}.$$

Proof. If we assume that $\det(I - zW(\xi_0)) = 0$ for one $\xi_0 \neq 1$, then there exists an eigenvector $\vec{e} \neq \vec{0}$ to z. Let us set

$$\vec{u}(\xi) := \frac{(1-\xi)^{1-\alpha}}{\xi - \xi_0}\vec{e} = \left(\frac{(1-\xi)^{1-\alpha} - (1-\xi_0)^{1-\alpha}}{\xi - \xi_0} + (1-\xi_0)^{1-\alpha} \frac{1}{\xi - \xi_0} \right) \vec{e} \quad .$$

Hence

$$\vec{g}(\xi) = \frac{1}{\xi - \xi_0} \left((1-\xi)^{1-\alpha} (I - zW(\xi)) - (1-\xi_0)^{1-\alpha} (I - zW(\xi_0)) \right) \vec{e} \quad .$$

Using lemma 3.1 we establish that the coefficient sequences of $\vec{g}(\xi)$ are in c_0. On the other hand, $(\vec{u}_n)_{n \in \mathbb{N}}$ is a divergent sequence. However, this contradicts the assumption $z \in S$.

It remains to prove $\det(I - zW(1)) \neq 0$ for all $z \in S$, equivalently $z \neq 0$ and $1/z$ is not an eigenvalue of $A(1)$. Choosing $g(t) = 1$, the condition $z \neq 0$ follows immediately. Let us suppose that $\det(I_{m-1} - zA(1)) = 0$. Hence there exists a vector $\vec{e}_{m-1} \neq \vec{0} \in \mathbb{C}^{m-1}$ with $(I_{m-1} - zA(1)) \cdot \vec{e}_{m-1} = \vec{0}$. If we set $e_m := \frac{-1}{\Gamma(1-\alpha)} \vec{a}_2^{\,t}(1) \cdot \vec{e}_{m-1}$ it follows that $\vec{e}(\xi) := \left(\vec{e}_{m-1}^{\,t}, (1-\xi)^{1-\alpha} e_m\right)^t$ is an eigenvector for the eigenvalue $1/z$ of $BW(\xi)B^{-1}$ with $\xi = 1$. We define $\vec{u}(\xi) := (1-\xi)^{-1} B^{-1} \vec{e}(\xi)$ and obtain that

$\vec{u}(\xi)$ has at least one component whose coefficient sequence is not in c_0. On the other hand, since $\frac{1}{z}$ is an eigenvalue of $BW(1)B^{-1}$, we have:

$$\vec{g}(\xi) = \frac{1}{1-\xi}\left[B^{-1}\left(I - zBW\left(\xi\right)B^{-1}\right)\vec{e}(\xi) - B^{-1}\left(I - zBW(1)B^{-1}\right)\vec{e}(1)\right] \quad .$$

The coefficient sequences of $B^{-1}\left(I - zBW\left(\xi\right)B^{-1}\right)\vec{e}(\xi)$ are in l^1. Lemma 3.1 yields that the coefficient sequences of $\vec{g}(\xi)$ are in c_0, which contradicts the assumption $z \in S$. ∎

If we consider bounded solutions, we can extend the proof of sufficiency (theorem 3.2) completely, but we have to restrict the above proof of necessity to $|\xi_0| < 1$. Later, it will be shown (lemma 5.5) that

$$\bar{S} = \{z \in \mathbb{C} \mid \det\left(I - zW\left(\xi\right)\right) \neq 0 \textit{ for all } \xi \in \mathbb{C} \textit{ with } |\xi| < 1\} \quad ,$$

where $\bar{S}$ denotes the closure of the stability region S. Hence we obtain for bounded functions

Corollary 3.6.

a.) Let g be bounded and $\lambda h^{1-\alpha} \in S$. Then the collocation approximation u is bounded.

b.) If the collocation approximation u is bounded for all bounded g, then $\lambda h^{1-\alpha} \in \bar{S}$.

4. Bounds for the stability domain and strong stability

The stability domain S of the collocation method for the Volterra integral equation (1.1) is characterized in terms of the inverses of the eigenvalues of $W\left(\xi\right)$. Since the first $m-1$ lines of the transformed matrix $BW\left(\xi\right)B^{-1}$ have absolutely convergent coefficients for all $|\xi| \leq 1$, $m-1$ eigenvalues can be estimated by Gerschgorin's theorem. The mth eigenvalue is an unbounded function of ξ, but we have to consider the reciprocal of the eigenvalues. For $R < 0$ we introduce the notion $\overline{B_R\left(y\right)} := \mathbb{C} \setminus B_{-R}\left(y\right)$ and for $R = \infty$ the notion $\overline{B_\infty\left(\infty\right)} := \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\}$. We can state that the function $f : z \to \frac{1}{z}$ is a bijective map from $\left\{z \in \mathbb{C} \cup \{\infty\} \mid \operatorname{Re} z \geq \frac{1}{2R}\right\}$ into $\overline{B_R\left(R\right)}$ for every $R \in \mathbb{R} \cup \{\infty\}$ and holomorphic except at $z = 0$. Hence we have $\mathbb{C} \setminus \left\{\frac{1}{z} \in \mathbb{C} \mid \operatorname{Re} z \geq \frac{1}{2R}\right\} = B_{-R}\left(R\right)$.

For estimating $\operatorname{Re} a_{mm}\left(\xi\right)$ and for later analysis we need the following result which is due to Erdélyi [1953].

Lemma 4.1. *Let Φ be defined for all $\xi \in \mathbb{C}$ with $|\xi| \le 1$, $s > 0$ and $r \neq 0, -1, -2, \ldots$ by $\Phi(\xi, s, r) := \sum\limits_{n=0}^{\infty} (n+r)^{-s} \xi^n$. Then Φ satisfies for $r > 0$, $|\xi| \le 1$, $\xi \neq 1$ and $\operatorname{Re} s > 0$ or for $r > 0$, $|\xi| \le 1$ and $\operatorname{Re} s > 1$*

$$\Phi(\xi, s, r) = \frac{1}{\Gamma(s)} \int\limits_0^{\infty} t^{s-1} e^{-(r-1)t} \frac{1}{e^t - \xi}\, dt \quad .$$

Furthermore, by the last representation, Φ is an analytical function with respect to ξ for $r > 0$ and $\operatorname{Re} s > 0$ on $\mathbb{C} \setminus \{\xi \in [1, \infty)\}$. If $\operatorname{Re} s < 1$ we have $\lim\limits_{\xi \to 1} (1-\xi)^{1-s} \Phi(\xi, s, r) = \Gamma(1-s)$.

As an additional remark, we note $\xi\ \Phi(\xi, \alpha, r+1) = \sum\limits_{j=1}^{\infty} (j+r)^{-\alpha} \xi^j$. By calculating the minimum of $\operatorname{Re}\left(\frac{\xi}{e^t - \xi}\right)$ for $|\xi| \le 1$, we obtain $\operatorname{Re} \xi\ \Phi(\xi, \alpha, r) \ \ge\ -\Phi(-1, \alpha, r)$.

Since we know that $a_{im}(\xi)$ is absolutely convergent for $i \neq m$ and $\xi \le 1$, we see that $\operatorname{Re} a_{mm}(\xi)$ has a lower bound in view of the following

Lemma 4.2.

$$\textit{Let} \qquad r(\xi) = \int\limits_0^{c_m} (c_m - v)^{-\alpha}\ dv + \sum_{j=1}^{\infty} \xi^j \int\limits_0^1 (j + c_m - v)^{-\alpha}\ dv$$

then $a_{mm}(\xi)$ can be written as $a_{mm}(\xi) = \sum\limits_{i=1}^{m-1} \mathbb{L}_{c_i} a_{im}(\xi) + r(\xi)$. Furthermore, for all $|\xi| \le 1$ we have $\operatorname{Re} r(\xi) \ge r(-1)$.

Using the theorem of Gerschgorin for estimating $\operatorname{Re}(z)$ of each eigenvalue z we can prove the following result.

Corollary 4.3. *There exist R_m^- and R_i , R_i^+ , R_i^- for $i = 1, \ldots, m-1$, which are all different from zero, so that*

$$S \supseteq B_{-R_m^-}(R_m^-) \cap \bigcap_{i=1}^{m-1} \left(B_{R_i}(0) \cup B_{-R_i^-}(R_i^-) \cup B_{R_i^+}(R_i^+) \right) \quad .$$

The stability domain is not empty and includes at least a circle $B_r(-r)$ with $r > 0$.

Adapting the ideas of Lubich [1986] and applying theorem 3.3 and the above corollary we obtain

Theorem 4.4. *The collocation method for the Volterra integral equation (1.1) is strongly stable.*

Corollary 4.3 yields only the existence of estimates. In Blank [1991] values for the estimates are determined by applying results for convex series, completely monotonic sequences and more detailed analysis of the function Φ.

5. Results for the boundary of the stability domain

5.1. Objectives: The main result of this section will be embodied in the theorem 5.4. In particular we have

$$\partial S \subseteq \left\{ \frac{1}{z} \in \mathbb{C} \mid z \textit{ is an eigenvalue of } W(\xi) \textit{ for some } \xi \textit{ with } |\xi| = 1 \right\} \quad . \tag{**}$$

Numerical experiments suggest that the identity instead of the inclusion is in general false.

5.2. Technical problems: We denote the closed complex plane as $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. The topology of $\mathbb{C}$ is extended to $\mathbb{C}_\infty$ by the identification of $\mathbb{C}_\infty$ with the sphere S^2 using stereographic projection.

The eigenvalues of $W(\xi)$ are the zeros of the characteristic polynomial

$$\wp(z,\xi) = z^m - A_1(\xi) z^{m-1} + \cdots + (-1)^m A_m(\xi) \quad .$$

The coefficients of $W(\xi)$ can be expressed in terms of Φ. Therefore, lemma 4.1 shows that the coefficients $A_j(\xi)$ of the characteristic polynomial $\wp(z,\xi)$ are holomorpic in $\mathbb{C} \setminus [1,\infty)$ and continuous in $\overline{B_1(0)}$ with respect to the $\mathbb{C}_\infty$−topology.

The following discussion is rather technical, and its inclusion (in the interest of rigour) merits some justification. To this end, we make the following remarks. If $m \neq 1$, the zeros of complex polynomials cannot be described by global continuous functions with respect to ξ in general. As an example consider $\wp(z,\xi) = z^2 - \xi$ for $\xi = e^{i\theta}$ with zero $e^{i\theta/2}$, which converges to i as $\theta \to \pi$ but converges to $-i$ as $\theta \to -\pi$. In consequence, it is necessary to construct Riemann surfaces. In order to construct a Riemann surface we are faced with the following questions: 1) For how many $\xi \in \overline{B_1(0)}$ has $\wp(.,\xi)$ zeros of order more than one and how are these ξ distributed? 2) Are they countable? 3) Are the functions of zeros holomorphic at the branch points? Because we also need results for $|\xi| = 1$ and $A_j(\xi)$ $(i = 1, \ldots, m)$ are not holomorphic in $[1, \infty)$, the construction of the surface is more complex.

Let us choose some $\varepsilon > 0$. We define $C_\delta := \{z \in \mathbb{C} \mid |z - x| < \delta \text{ for some } x \in [1, 1+\varepsilon]\}$ and $B := B_{1+\varepsilon}(0) \setminus [1, 1+\varepsilon)$. We know (Grauert, Fritzsche [1974]): *The ring of normed polynomials over functions, which are holomorphic in a domain, is factorial.* Hence there exists a unique prime factorization of $\wp$ into $\wp = p_1 \cdot p_2 \cdot \ldots \cdot p_s$, where p_i are irreducible polynomials in the ring of normed polynomial over holomorphic functions on B. Furthermore they are continuous in $\overline{B_1(0)}$ with respect to the $\mathbb{C}_\infty$−topology.

Our construction of a Riemann surface above $\overline{B_1(0)}$ with the desired properties proceeds in the following steps: Firstly we consider $p = p_i$ and construct a Riemann surface R_δ above $B \setminus C_\delta$. Secondly we extend this surface to a Riemann surface R above B and note the properties of the functions of zeros on the extension to R' above $B \cup \{1\}$. After the restriction to the surfaces above $\overline{B_1(0)}$ for $p = p_i$ we finish with the conclusion for $\wp$ and summarize the results in theorem 5.3.

1.: We consider $p := p_i$. Associating $\Delta_p(\xi)$ with the discriminant of $p(z, .)$ at the point ξ, we state that $p(z, .)$ has only simple zeros at ξ if, and only if, $\Delta_p(\xi) \neq 0$ (Grauert, Fritzsche [1974]). Furthermore we can show (applying the identity principle) the following result, which answers question 1) and 2).

Lemma 5.1. *$D_p := \{\xi \in B \mid \Delta_p(\xi) = 0\}$ has at most finitely many points in $\overline{B \setminus C_\delta}$ for any $\delta > 0$ and consists of countably isolated points in B, which accumulate at most at $[1, 1+\varepsilon]$.*

Since p is a normed polynomial without repeated factors, we can use the following theorem (Grauert, Fritzsche [1974, p.96]) in order to construct the Riemann surface for p.

Theorem 5.2. *Let p be a normed polynomial without repeated factors. Then for every $\xi_0 \in B \setminus D_p$ there exists a neighborhood $U(\xi_0) \subseteq B \setminus D_p$ and holomorphic functions $f_1, \cdots, f_l$ in $U(\xi_0)$ with $f_\mu(\xi) \neq f_\nu(\xi)$ for $\mu \neq \nu$ and for all $\xi \in U(\xi_0)$, such that $p(z, \xi) = (z - f_1(\xi)) \cdot \ldots \cdot (z - f_l(\xi))$ for all $\xi \in U(\xi_0)$. Above any point $\xi \in D_p$ we have less than l distinct zeros of $p(., \xi)$, i.e. the points of D_p are branch points.*

Returning to our problem we address question 3) and construct the appropriate Riemann surface above $B \setminus C_\delta$.

p has exactly l zeros at each point $\xi \in B$. For $i = 1, \cdots, l$ B_i denotes a copy of the domain B. On each B_i we denote the points ξ_n of D_p as $\xi_n^{(i)}$. Furthermore, for every point $\xi_n \in D_p$ there exists a neighborhood $U(\xi_n) \subseteq B$, which includes no other point of D_p. Similar to the construction of the Riemann surface for roots, we obtain using the identity theorem, theorem 6.2 and the continuity of $W(\xi)$, that z is holomorphic in $U\left(\xi_n^{(i)}\right) \setminus \left\{\xi_n^{(i)}\right\}$ and continuous at $\xi_n^{(i)}$ of each B_i. Here the continuity is associated with the $\mathbb{C}-$ or $\mathbb{C}_\infty-$topology depending on the value at $\xi_n^{(i)}$.

In $B \setminus C_\delta$ only finitely many points of D_p exist. Hence for each $\xi_n \in D_p \cap \{B \setminus C_\delta\}$ we find a point $x_n \in (\partial B) \setminus C_\delta$, such that the cuts $\sigma_n = \overline{x_n \xi_n}$ do not intersect or touch each other. Now cut $B \setminus C_\delta$ along all lines σ_n. Then $B' := B \setminus \left\{ C_\delta \cup \bigcup_{\xi_n \in D_p} \sigma_n \right\}$ is a domain. Recalling the identity theorem and theorem 6.2, we can continue each f_i such that f_i is holomorphic in this domain. Therefore there exist Riemann surfaces R_δ above $B \setminus C_\delta$ such that the function of zeros z is holomorphic in R_δ except at the branch points in D_p. At these branch points, z is continuous at least with respect to the $\mathbb{C}_\infty-$topology.

Furthermore $\overline{B \setminus C_\delta}$ is compact and all A_j are holomorphic in $\mathbb{C} \setminus [1, \infty)$. Therefore all A_j are bounded in $\overline{B \setminus C_\delta}$. The theorem of Gerschgorin shows that z is bounded in R_δ and continuous with respect to the $\mathbb{C}-$topology. Since z is holomorphic except at the isolated points $\xi_n^{(i)}$, the Riemann continuation theorem implies that z is holomorphically extendable to the whole surface R_δ.

2.: In the following we obtain a Riemann surface above $\overline{B_1(0)}$:

There are only finitely many elements of D_p in $B \setminus C_\delta$ for all $\delta > 0$, hence it is possible to construct R_{δ_1} for $0 < \delta_1 < \delta_2$ so that the reduction of R_{δ_1} to the copies above $B \setminus C_{\delta_2}$ is identical to R_{δ_2}. Since $B = \bigcup_{n \in \mathbb{N}} (B \setminus C_{1/n})$, we define the Riemann surface R above B as the union of the constructed Riemann surfaces R_δ $(\delta > 0)$. For every point x in B_i of R above B we find $\delta > 0$, where $x \in R_\delta$ and z is holomorphic at x. Hence z is holomorphic in R. Furthermore, in the ring of normed polynomials over functions, which are holomorphic in B, p is an irreducible polynomial. This implies that all B_i of R are connected.

Now we extend the Riemann surface R with the value 1 in each B_i to R'. The matrix $W(\xi)$ is continuous at $\xi = 1$ with respect to the $\mathbb{C}_\infty-$topology and the eigenvalues, except one, are bounded (corollary 4.3). Thus, we see by the uniqueness of the function of zeros z and its description, that, except for one B_s, z has to be continuous with respect to the $\mathbb{C}-$topology in $B_i \cup \{1\}$ at 1. In $B_s \cup \{1\}$, z is continuous at 1 with respect to the $\mathbb{C}_\infty-$topology.

We finish the construction of the Riemann surfaces with the restriction of R' above $B_{1+\varepsilon}(0) \backslash (1, 1+\varepsilon)$ to the Riemann surfaces above $\overline{B_1(0)}$. Since the polynomial p is irreducible in the ring of the normed polynomials over holomorphic functions on B and not on $\overline{B_1(0)}$, the surface R' can fall into several surfaces R_j, and z into functions z_j on R_j.

If we now return to $\wp$, corollary 4.3 shows that there exists only one unbounded function of zeros z_l of $\wp$, i.e. $z_l(\xi_0) = \infty$ for $\xi_0 \in R_l$ above 1. Considering now all zeros of $\wp$, i.e. all eigenvalues of $W(\xi)$, we can conclude the first essential step of the proof of (**).

Theorem 5.3. *The set M of eigenvalues of $W(\xi)$ for $\xi \in \overline{B_1(0)}$ is determined by functions $z_1, \cdots, z_l$ on compact Riemann surfaces $R_1, \cdots, R_l$ above $\overline{B_1(0)}$ with $l \leq m$, i.e.*

$$M = \bigcup_{i=1}^{l} z_i(R_i) \quad .$$

These functions z_i are holomorphic in R_i except at the points above $1 \in \overline{B_1(0)}$.

The functions $z_1, \ldots, z_{l-1}$ are continuous in complete R_i with respect to the $\mathbb{C}-$topology. Furthermore, $z_l(\xi_0) = \infty$ for only one $\xi_0 \in R_l$ above $1 \in \overline{B_1(0)}$. z_l is continuous in complete R_l with respect to the $\mathbb{C}_\infty-$topology, and also with respect to the $\mathbb{C}-$topology except at ξ_0.

5.3. Substantive results:

Theorem 5.4.

Let $f(z) := 1/z$. Then the stability domain of the collocation method satisfies

$$S = \mathbb{C} \setminus f(M) = \bigcap_{i=1}^{l} \mathbb{C} \setminus f(z_i(R_i)) \quad \textit{and}$$

$$\partial S \subseteq \left\{ \frac{1}{z} \in \mathbb{C} \mid z \textit{ is an eigenvalue of } W(\xi) \textit{ for } \xi \textit{ with } |\xi| = 1 \right\} \quad .$$

Furthermore we have:

1.) The boundary of S is given by at most m closed curves. These curves are parts of $f(z_i(\partial R_i))$. For all $i = 1, \ldots, m$ $f \circ z_i$ is analytic on ∂R_i except at the points above $1 \in \partial B_1(0)$ and at the zeros of z_i.

2.) $0 \in \partial S$ and there exists only one boundary curve of S which runs through 0.

3.) S is open and symmetric with respect to the real axis.

Proof. The continuity of z_i on R_i yields $z_i(R_i)$ is compact with respect to the $\mathbb{C}_\infty-$topology. This implies $z_i(R_i) \supseteq \partial z_i(R_i)$. Moreover, if z_i is not constant, the theorem on preservation of regions on connected manifolds for holomorphic functions indicates that $z_i\left(\mathring{R}_i\right)$ is a region. If z_i is constant, $z_i(R_i)$ is a point. Hence M consists of maximal m connected sets and $z_i\left(\mathring{R}_i\right) \cap \partial z_i(R_i) = \emptyset$. Thus we can state that $z_i(\partial R_i) \supseteq \partial z_i(R_i)$.

Furthermore, $\partial M \subseteq \partial\left(\bigcup_{i=1}^{l} z_i(R_i)\right) \subseteq \bigcup_{i=1}^{l} z_i(\partial R_i)$ and z_i represents on ∂R_i the eigenvalues of $W(\xi)$ for $\xi \in \partial B_1(0)$. Since f is biholomorphic in $\mathbb{C}_\infty$ we obtain the same for $f \circ z_i$. Hence

$$\partial S = \partial f(M) \cap \mathbb{C} = f(\partial M) \cap \mathbb{C} \subseteq \left\{\frac{1}{z} \in \mathbb{C} \mid z \text{ is an eigenvalue of } W(\xi) \text{ for } \xi \text{ with } |\xi| = 1\right\}. \quad \blacksquare$$

Using this theorem we can now prove the following lemma, which was cited in section 3.

Lemma 5.5. *Let $\bar{S}$ denote the closure of S. Then*

$$T := \{z \in \mathbb{C} \mid \det(I - zW(\xi)) \neq 0 \textit{ for all } \xi \in \mathbb{C} \textit{ with } |\xi| < 1\} = \bar{S} \quad .$$

Proof. Let $\mathring{f}(z_i(R_i))$ be the interior of $f(z_i(R_i))$. Then we have

$$T = \bigcap_{i=1}^{l} \mathbb{C} \setminus f\left(z_i\left(\mathring{R}_i\right)\right) \quad \text{and} \quad \overline{S} = \bigcap_{i=1}^{l} \mathbb{C} \setminus \mathring{f}(z_i(R_i)) \ .$$

Using $f\left(z_i\left(\mathring{R}_i\right)\right) \cap \partial f(z_i(R_i)) = \emptyset$ we find $f\left(z_i\left(\mathring{R}_i\right)\right) \subseteq \mathring{f}(z_i(R_i))$. On the other hand, since the inverse image of open sets for continuous functions are open sets, $(f \circ z_i)^{-1}\left(\mathring{f}(z_i(R_i))\right)$ is open und hence in $\mathring{R}_i$. Now we can conclude

$$\mathring{f}(z_i(R_i)) \subseteq f\left(z_i\left(\mathring{R}_i\right)\right) \Rightarrow \mathring{f}(z_i(R_i)) = f\left(z_i\left(\mathring{R}_i\right)\right).$$

This immediately implies the assertion. $\blacksquare$

The discussion involving Riemann surfaces has an application in other areas, notably in the discussion of stability of implicit Runge–Kutta methods for ordinary differential equations. For example in his discussion, Butcher [1987] p.239, presents the arguments for explicit, rather than for implicit Runge–Kutta methods. When determining the boundary of the stability regions for implicit Runge–Kutta methods one can use the arguments developed above. This does not appear to have been remarked upon in the open literature.

5.4. Practical significance: Now we have established the basis to determine the stability domain numerically. We plot $f(z_i(\partial R_i))$ for all i by calculating the set

$$I := \{1/z \in \mathbb{C} \mid z \text{ is eigenvalue of } W(\xi) \text{ for a } \xi \text{ with } |\xi| = 1\} \quad ,$$

but we must pay attention to the fact that the boundary of S does not necessarily consist of all parts of these curves. At least, the smallest domain S' which includes $B_\delta(-\delta)$ for any small $\delta > 0$ and is bounded by parts of $f(z_i(\partial R_i))$, belongs to the stability domain S. Depending on the features of $f(z_i(\partial R_i))$ we can derive a more accurate description of S using the fact that $f\left(z_i\left(\mathring{R}_i\right)\right)$ is a domain and $f \circ z$ is continuous. If we cannot decide which regions belong to S, we can plot

$$I_\varepsilon := \{1/z \in \mathbb{C} \mid z \text{ is eigenvalue of } W(\xi) \text{ for a } \xi \text{ with } |\xi| = 1 - \varepsilon\}$$

for any small $\varepsilon > 0$. Since $I_\varepsilon \in f(M)$ and I_ε converges for $\varepsilon \to 0$ to I, we see which regions belong to S and which belong to $f(M)$.

6. Examples

If we consider the collocation approximation of Volterra integral equation (1.1) –given by (2.1)– with polynomial splines of degree zero, we can state the following results (cf. Blank[1991]): for $c_1 \geq \frac{1}{2}$ the method is $A\left(\frac{\pi}{2}\right)$–stable; if $c_1 = 1$ we have at least $S \supseteq \mathbb{C} \setminus \overline{B_1(1)}$; if $c_1 \neq 0$, then the stability domain converges to $\mathbb{C} \setminus \{0\}$ as $\alpha \to 1$; and if $c_1 = 0$, then the stability domain converges to an empty set as $\alpha \to 1$.

If we choose the collocation solution $u \in S_m(Z)$ with $c_1 = 0$ and $c_m = 1$, then u is continuous. Furthermore, we can reduce the m–dimensional collocation equation to an $(m-1)$–dimensional equation.

Then we can obtain a characterization of the stability domain involving one less eigenvalue. In the simplest case, $m = 2$, we obtain that the method is $A\left(\frac{\pi}{2}\right)-$stable and the stability domain converges to $\mathbb{C} \setminus \{0\}$ as $\alpha \to 1$. In general we can show for $u \in S_2(Z)$ that: if $\alpha = 0$, the method is $A-$stable if, and only if, $c_1 + c_2 \geq 1$. For details and more stability results we refer to Blank [1991].

In the following figures we illustrate some numerical examples. We evaluate the finite series $\sum_{j=0}^{N} BW_j B^{-1}\xi^j$ for ξ with $|\xi| = 1$, Im $(\xi) \geq 0$ and equally divided imaginary parts, using fixed N and then determine the eigenvalues. The maximal number N of summands would have to be very large to obtain a good approximation of the series, when α is close to zero. It is obvious that this may create large rounding errors. Numerical experiments have shown that $W(\xi)$ can be determined relatively accurately if α is larger than 0.25. The inverses of these eigenvalues were interpolated for ξ on the unit circle by cubic splines, such that the curve of the boundary is smooth except $\xi = 1$ and is continuous at $\xi = 1$. The marked points in the figures are the computed values. If not mentioned otherwise, the curves for increasing parameter intersect first the negative axis (if they do) and then the positive axis in decreasing real values. The stability domains always include at least a disc $B_\delta(-\delta)$ for some $\delta > 0$.

For m=1:

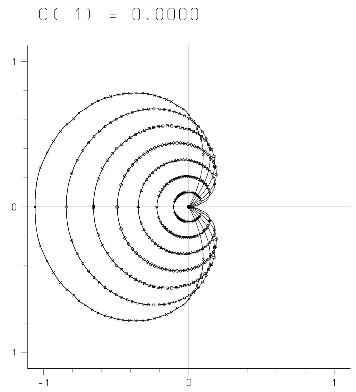

Figure 1

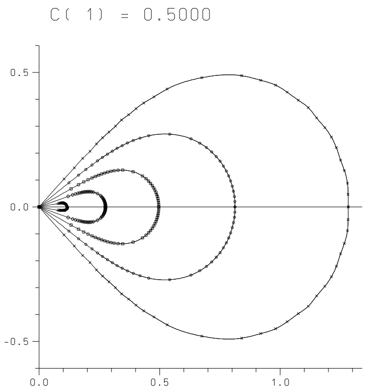

Figure 2

In figure 1, the boundaries for increasing $\alpha = 0.3\ ,\ 0.4\ ,\ldots,\ 0.9$ are exceptionally represented by the curves, which intersect the negative real axis in increasing values. The stability domains lie within the boundaries. We observe that the stability domain is not monotonic with respect to α, i.e. for any α the stability domain does not include the stability domains of smaller α. In figure 2, the boundaries for $\alpha = 0.5, 0.6, \ldots, 0.9$ are plotted. The stability domains lie outside the bounded areas. In contrast to the case $c_1 = 0$ the method is monotone for $c_1 = 1/2$ with respect to α. The numerical solutions indicate that for the choice of $c_1 = 1/2$ and $c_1 = 1$ the method is $A-$stable, not only for $\alpha = 0$, but for all α.

The plots for $\alpha = 0.25$ and $\alpha = 0.5$ are similar to the following figures 3 and 4. Figure 3 represents the stability boundaries for $c_1 = 0.0, \ldots, 0.5$ and figure 4 shows the boundaries for $c_1 = 0.4, \ldots, 1.0$. The stability boundaries for the exact solutions are plotted as well (the lines). It seems that the stability domain behaves monotonically with respect to c_1.

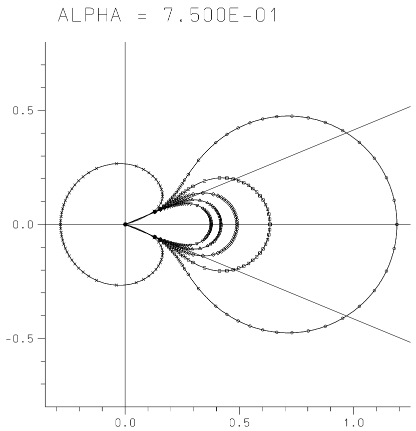

Figure 3

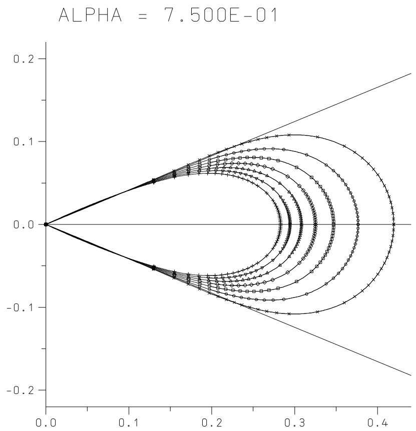

Figure 4

The numerically calculated stability domains for the choice of collocation points as Lobatto-, Radau- and Gauß-Legendre-points in the two- and three-dimensional cases indicate that the stability domain behaves monotonically with respect to α and converges to $\mathbb{C} \setminus \{0\}$ as $\alpha \to 1$. Furthermore, they are $A-$stable, as we have proved theoretically for $\alpha = 0$. Below we show the plots for the choice of Lobatto-points. The others look similar. The curves represent the stability boundaries for $\alpha = 0.5, 0.6, 0.7, 0.8$ and 0.9.

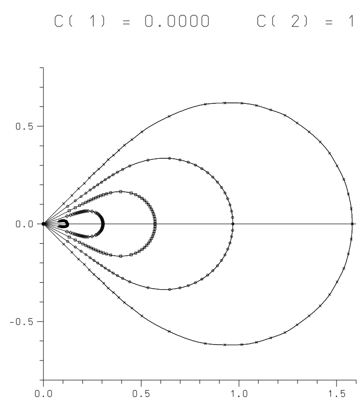

Figure 5

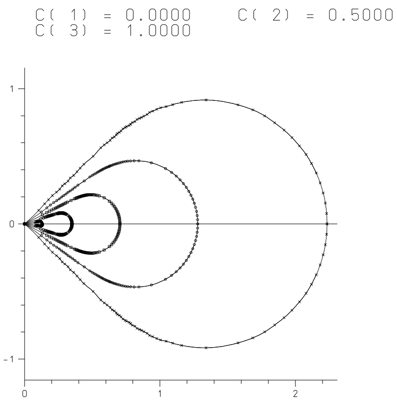

Figure 6

Figures 7 – 9 illustrate the stability domains for fixed c_2 and increasing c_1 for $m = 2$, $\alpha = 0.5$. We observe that in figure 7 ($c_2 = 0.2$), the stability domain for $c_1 = 0.1$ does not include that for $c_1 = 0$. In figure 9 ($c_2 = 1$), we notice that the stability boundaries are not monotone with respect to c_1, since for decreasing order of intersections with the positive axis, the curves represent the boundaries for $c_1 = 0.1$, $c_1 = 0$, $c_1 = 0.2$, $c_1 = 0.3$ and so on. This behaviour appears for all tested $\alpha \neq 0$ and $c_2 = 1$.

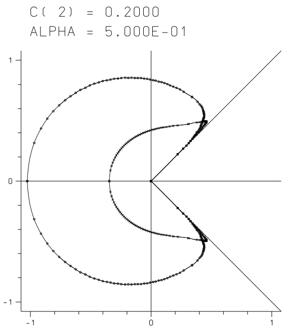

Figure 7

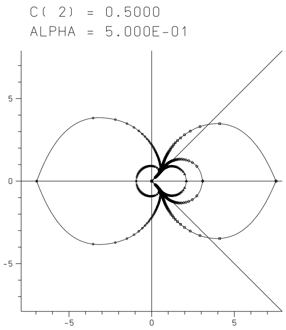

Figure 8

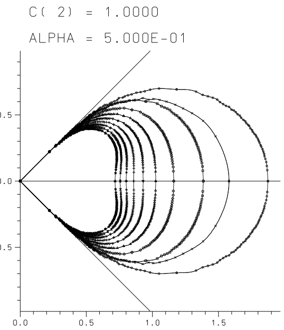

Figure 9

The following examples depict different forms of the stability domain. We see that the stability domain can be given by the intersection of two domains and that the form of the stability boundary can suddenly jump to another feature by varying c_1. Such phenomena also occur for other values of α.

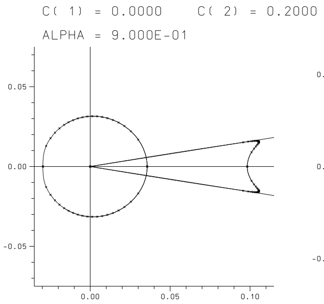

Figure 10

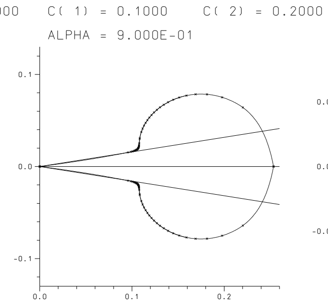

Figure 11

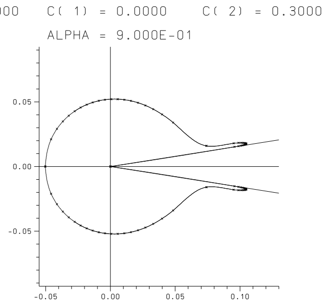

Figure 12

If we consider splines of second degree ($m = 3$), we cannot recognize essentially new effects for the stability domains. For $\alpha = 0.5$ we choose a few examples, which have a rarely occuring shape, or where the form of the stability boundaries change suddenly in successive examples (c_2 increases). Moreover, as introduced in section 5, we illustrate by plotting the inverses of the eigenvalues also for $|\xi| = 1 - \epsilon$, how we can decide which domain belongs to the stability domain. Here the stability domains are represented by the shaded regions. In figure 16 for $c_1 = 0$ and $c_3 = 0.8$, the boundaries of the stability domains are plotted for increasing c_2. Most stability domains vary in this manner.

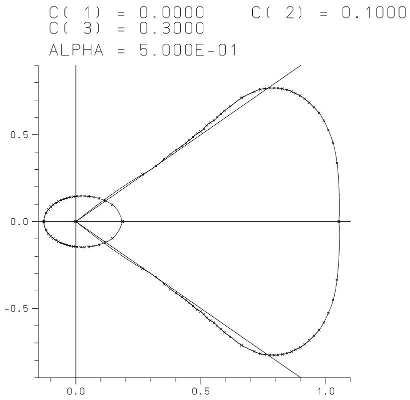

Figure 13

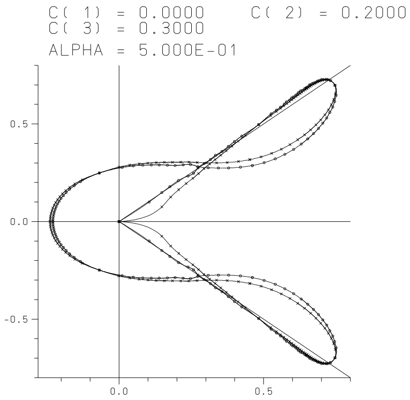

Figure 14

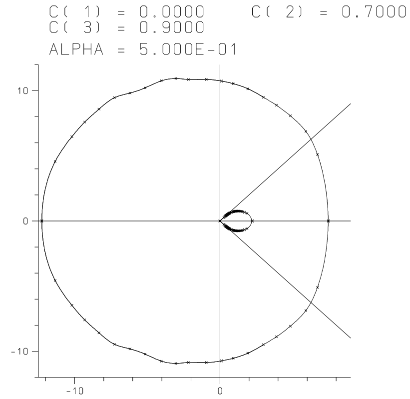

Figure 15

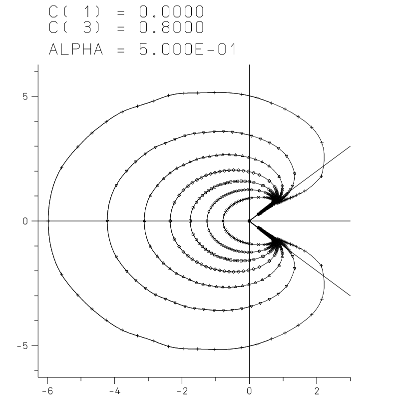

Figure 16

The numerical examples suggest that the stability domain of any α includes the stability domains of smaller α. However, this is not true for the choice $m = 1$ and $c_1 = 0$. There arises the question: Is this the only exception?

Results for monotone behaviour of increasing collocation parameters could not be obtained. Large differences exist between the forms of stability domains especially with the change from one to two boundaries.

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