Stability results for collocation methods for Volterra integral equations

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Abstract. Stability behavior is investigated for polynomial spline collocation applied to Volterra integral equations with special emphasis on a weakly singular test equation. The characterization of the corresponding stability domain is related to stability results for solutions of finite recursion relations, as they occur for methods applied to ordinary differential equations. Estimates of the stability domains are established and, as a consequence, conditions for $A(\pi/2)$-stability are obtained. Exploiting these estimates, collocation approximations with constant, linear, and continuous splines are considered in more depth and several numerical illustrations are presented.

Subject Classifications. AMS: 65R20, 45D05, 45L10.

Keywords: Stability, Volterra integral equation, weakly singular kernel, collocation method.

1. Introduction

Numerical methods for Abel integral equations have been considered by many authors and we refer to Chapters 6 and 7 in Brunner and van der Houwen [1986]. In particular Brunner introduced the collocation method with polynomial splines and analyzed its convergence. We investigate the asymptotic stability behavior of this method applied to the Abel integral equation of the second kind

$$y(t) \;=\; g(t) + \lambda \int_0^t (t-s)^{-\alpha} y(s)\, ds\ , \quad t \geq 0 \tag{1.1}$$

where $\lambda \in \mathbb{C}$ and $\alpha \in [0,1)$. Lubich [1986] considered this test equation for the stability analysis of fractional linear multistep methods. The restriction to constant perturbations ($g(t) \equiv c$), which is common in the literature for regular kernels (e.g., Bellen et al. [1990]), and used by Brunner et al. [1991] for c-methods, is abandoned. In particular, for $\alpha = 0$ it has been shown (Blank [1991]) that the stability domains for $g(t) \equiv c$ and $\lim\limits_{t \to \infty} g(t) = c$ are identical. A more general class of test equations —namely with nonlinear kernels— has been considered by Eggermont to study the existence and uniqueness of the numerical solution u in $L^p(\mathbb{R}^+)$ for the trapezoidal method [1988] and the question of $u \in L^\infty(\mathbb{R}^+)$ for the closely related Galerkin approximation [1989]. In contrast, we address the question: Suppose the function g tends to a constant, when does the collocation approximation u converge to zero as t tends to infinity?

The characterization of the stability domain for the collocation method leads to the analysis of

$$\det\left(I - \lambda h^{1-\alpha} W(\xi)\right) \neq 0 \text{ for all } |\xi| \leq 1\ ,$$

where $W(\xi) = \sum\limits_{n=0}^{\infty} W_n \xi^n$, and the W_n depend on α and on the collocation parameters c_i $(i = 1, \ldots, m)$. We observe that this characterization generalizes the known result for the stability domains of solutions of finite recursion relations, which uses the zeros of the characteristic polynomial (Section 2). The major difficulty in exploiting the description of the stability domain arises from the power series $W(\xi)$ which has even singularities at $\xi = 1$. The first step is to reduce the occurence of the singularities in each coefficient of $W(\xi)$ to a singularity in one coefficient only. In particular, we apply a matrix transformation and examine the behavior of its coefficients (Section 3). This enables us to establish estimates of the eigenvalues of $W(\xi)$. As a consequence, we obtain conditions for $A(\pi/2)$-stability and criteria to choose h for $\lambda \in \mathbb{C}^-$ such that the method is stable (Section 4). Moreover, we consider the collocation method with piecewise constants, linear splines, and continuous splines in more depth, and illustrate the results by numerical examples (Section 5).

2. Collocation approximation, its known stability results, and comparison with finite recursion relations

2.1. Collocation approximation: Assume Z to be a uniform partition of $\mathbb{R}_+$ and let the m collocation parameters be denoted by $0 \leq c_1 < \cdots < c_m \leq 1$. Their projections on $[t_n, t_{n+1}]$ determine the collocation points $t_n + c_i h$ $(i = 1, .., m)$. The polynomial spline collocation yields a piecewise polynomial u of degree $m-1$, which satisfies the Volterra integral equation at the collocation points. More precisely, u is an element of the spline space $S_{m-1}(Z) = \left\{ u \middle| u|_{[t_n, t_{n+1})} \in \Pi_{m-1} \right\}$, and the m free coefficients of u on the subinterval $[t_n, t_{n+1}]$ are given by

$$u(t_n + c_i h) = g(t_n + c_i h) + \lambda \int_0^{t_n + c_i h} (t_n + c_i h - s)^{-\alpha} \, u(s) \, ds \ , \quad (i = 1, \cdots, m) \ . \tag{2.1}$$

(For further details and convergence results see Brunner and van der Houwen [1986]). Throughout this paper we define:

the Lagrange polynomials $L_{c_l}(v) := \prod_{\substack{k=1 \\ k \neq l}}^{m} (v - c_k)/(c_l - c_k)$,

the weights $w_0(c_i, c_l) := \int_0^{c_i} (c_i - v)^{-\alpha} L_{c_l}(v) \ dv$

and $w_j(c_i, c_l) := \int_0^1 (j + c_i - v)^{-\alpha} L_{c_l}(v) \ dv \quad \text{for } j \geq 1$.

With the Lagrange representation of u in each subinterval and

$$\vec{u}_n = \Big(u(t_n + c_i h)\Big)_{i=1,\cdots,m} \quad , \quad \vec{g}_n = \Big(g(t_n + c_i h)\Big)_{i=1,\cdots,m} \quad \text{and} \quad W_j = \Big(w_j\,(c_i, c_l)\Big)_{\substack{i=1,\cdots,m\\ l=1,\cdots,m}}$$

we obtain the infinite recursion relation $\vec{u}_n = \vec{g}_n + \lambda h^{1-\alpha} \sum_{j=0}^{n} W_{n-j}\vec{u}_j$ for $n \geq 0$. Due to the fact that sequences can be identified with their formal power series ($\vec{u}(\xi) := \sum_{n=0}^{\infty} \vec{u}_n \xi^n$) the sequence $(\vec{u}_j)_{j\in I\!N_0}$ is determined by

$$\vec{u}(\xi) = \vec{g}(\xi) + \lambda h^{1-\alpha} W(\xi) \cdot \vec{u}(\xi) \quad . \tag{2.2}$$

These power series are an important tool for obtaining properties of the sequences of coefficients.

Before we introduce the stability concept for the numerical method we recall the theorem on the analytical solution of (1.1) (Lubich [1986], Sauren [1993]).

Theorem 2.1. *Let $g \in C\,([0,\infty))$. Then there exists a unique continuous solution y to* (1.1). *Furthermore, y is stable (i.e., y tends to zero as $t \to \infty$) for all g with finite limit as $t \to \infty$ if and only if the parameter λ satisfies the condition*

$$|\arg(\lambda) - \pi| < \frac{1}{2}(1+\alpha)\pi \quad .$$

The asymptotic stability concepts of numerical methods for ordinary differential equations are extended below to the Volterra integral equation (1.1). However, we shall not assume perturbations at the initial value only, but on the whole timescale $I\!R^+$ by perturbing the function g at any time.

Definition 2.2.

The stability domain S of the collocation method (2.1) for the Volterra integral equation (1.1) consists of all $\lambda h^{1-\alpha} \in \mathbb{C}$, for which the numerical solution $u(t)$ converges to 0 as $t \to \infty$, whenever $\lim_{t\to\infty} g(t) = const.$. The method is said to be $A(\theta)$-stable if S includes the region $|\arg(z) - \pi| < \theta$. Furthermore, the method is said to be A-stable if it includes the stability domain of the Volterra integral equation (1.1).

2.2. Stability results and comparison with finite difference equations: While infinite difference equations occur in general for numerical methods applied to integral equations and reflect the integral from 0 to t, finite difference equations arise from numerical methods applied to differential equations and reflect the local dependence of the derivative on t. For infinite difference equations resulting from the introduced collocation method, the following characterization of the stability domain was obtained in Blank [1993] using the Paley–Wiener inversion theorem. Similar results for further infinite recursion relations can be found in Lubich [1983, 1986].

Theorem 2.3. *The stability domain S of the collocation method with polynomial splines on a uniform partition for the Volterra integral equation (1.1) is given by*

$$S = \mathbb{C} \setminus \left\{ \lambda h^{1-\alpha} \middle| \frac{1}{\lambda h^{1-\alpha}} \textit{ is an eigenvalue of } W(\xi) \textit{ for a } \xi \in \mathbb{C} \textit{ with } |\xi| \leq 1 \right\}$$

$$= \left\{ z \in \mathbb{C} \,\middle|\, \det\left(I - z \sum_{j=0}^{\infty} W_j \nu^{-j} \right) = 0 \Rightarrow \nu < 1 \right\} \quad .$$

For finite difference equations $\sum_{i=0}^{k} M_i(z) \vec{u}_{n-i} = \vec{w_n} (n \geq k)$ the zeros of the characteristic polynomial determine the stability domain by

$$\left\{ z \in \mathbb{C} \,\middle|\, \det\left(\sum_{i=0}^{k} M_i(z) \xi^{k-i} \right) = 0 \Rightarrow |\xi| < 1 \right\} \quad .$$

But, despite the different approach analyzing the stability domain for finite and infinite difference equations, it is evident that the characterizations of the stability domains correspond to each other. The Routh Hurwitz criterion (Marden [1966]) can be used to prove whether the method is $A(\pi/2)$–stable for finite recursion relations only. For the introduced collocation approximation applied to the Abel integral equation (1.1), conditions for $A(\pi/2)$–stability are developed in Section 4. Furthermore, the following result (Blank [1993]) provides a rigorous description of the boundary of the stability domain S and is the theoretical foundation for the numerical evaluation and analysis of the stability domain in Section 5.

Theorem 2.4.

$$\partial S \subseteq \left\{ \frac{1}{z} \in \mathbb{C} \,\middle|\, z \textit{ is an eigenvalue of } W(\xi) \textit{ for some } \xi \textit{ with } |\xi| = 1 \right\} .$$

(The identity instead of the inclusion is in general false.) Furthermore we have: S is open, symmetric with respect to the real axis and contains at least a ball $B_\delta(-\delta)$ with sufficiently small $\delta > 0$. ∂S is given by at most m closed curves of which exactly one includes 0. The curves are given by parts of the reciprocals of the eigenvalue functions of $W(\xi)$ for $|\xi| = 1$, which are continuous with respect to the $\mathbb{C}_\infty$-topology on a Riemann manifold above $\overline{B_1(0)}$ and holomorphic above $\overline{B_1(0)} \setminus \{1\}$.

Due to the above, ∂S can by plotted by calculating the eigenvalues of $W(\xi)$ for $|\xi| = 1$. Then it is clear which regions belong to S. Recalling the numerical methods with finite recusion relations for which each $M_i(z)$ is a polynomial in z and $M_k(z)$ is the one with maximal power in z, the above theorem stays valid, i.e.,

$$\partial S \subseteq \left\{ z \in \mathbb{C} \,\middle|\, \det\left(\sum_{i=0}^{k} M_i(z)\xi^{k-i}\right) = 0 \quad \textit{for some } \xi \textit{ with } |\xi| = 1 \right\} .$$

3. Matrix transformation

The fundamental basis for all further analysis of the eigenvalues is the following matrix transformation $BW(\xi)B^{-1}$. The problem, that all coefficients of the matrix $W(\xi)$ are unbounded series, is reduced to one coefficient, which is an unbounded series. Basically the ideas for the choice of the matrix transformation are: the Lagrange polynomials add up to 1 and the singularity of $w(c_i, c_l)(\xi)$ depends only on c_l, since $w(c_i, c_l)(\xi) = (1-\xi)^{\alpha-1}\Gamma(1-\alpha)\int_0^1 L_{c_l}(v)\ dv + e(c_i, c_l)(\xi)$ and $e(c_i, c_l)(\xi)$ has a coefficient sequence in ℓ^1. This can be obtained by rewriting $w(c_i, c_l)(\xi)$ in terms of $j^{-\alpha}$, using sum formulas for $(1-\xi)^s$ and the fact that $\{j^{-s}\}_{j\in\mathbb{N}} \in \ell^1$ if $s > 1$ (see Erdélyi [1953]). Therefore we know that $w(c_i, c_l)(\xi) - w(c_m, c_l)(\xi)$ and $w(c_i, c_l)(\xi) - \int_0^1 L_{c_l}(v)\ dv \sum_{k=1}^{m} w(c_i, c_k)(\xi)$ have coefficient sequences in ℓ^1. We set $\mathbb{L}_{c_l} = \int_0^1 L_{c_l}(v)\ dv$

and define

$$B := \begin{pmatrix} 1 & & & & & -1 \\ & \cdot & & 0 & & \cdot \\ & & \cdot & & & \cdot \\ & 0 & & \cdot & & \cdot \\ & & & & 1 & -1 \\ \mathbb{L}_{c_1} & \cdot & \cdot & \cdot & \cdot & \mathbb{L}_{c_m} \end{pmatrix} \in \mathbb{R}^{m\times m} .$$

The inverse matrix is

$$B^{-1} = \begin{pmatrix} 1-\mathbb{L}_{c_1} & -\mathbb{L}_{c_2} & \cdot & \cdot & -\mathbb{L}_{c_{m-1}} & 1 \\ -\mathbb{L}_{c_1} & 1-\mathbb{L}_{c_2} & \cdot & & \cdot & 1 \\ \cdot & -\mathbb{L}_{c_2} & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & -\mathbb{L}_{c_{m-1}} & \cdot \\ \cdot & & \cdot & \cdot & 1-\mathbb{L}_{c_{m-1}} & \cdot \\ -\mathbb{L}_{c_1} & \cdot & \cdot & \cdot & -\mathbb{L}_{c_{m-1}} & 1 \end{pmatrix} .$$

Taking into account the analysis of the function $\Phi(\xi, s, r) := \sum\limits_{n=0}^{\infty} (n+r)^{-s}\, \xi^n$ (see Erdélyi [1953]), which is defined for all $\xi \in \mathbb{C}$ with $|\xi| \le 1$, $s > 0$ and $r \neq 0, -1, -2, \ldots$, we obtain the following theorem.

Theorem 3.1. *Let*

$$BW(\xi)\, B^{-1} = \begin{pmatrix} A(\xi) & \vec{a_1}(\xi) \\ & \\ \vec{a_2}^{\,t}(\xi) & a_{mm}(\xi) \end{pmatrix} .$$

Then $A(\xi) = \Big(a_{is}(\xi)\Big)_{\substack{i=1,\cdots,m-1\\ s=1,\cdots,m-1}}$, $\vec{a_1}(\xi) = (a_{im}(\xi))_{i=1,\cdots,m-1}$ and $\vec{a_2}(\xi) = (a_{ms}(\xi))_{s=1,\cdots,m-1}$ *have coefficient sequences in* ℓ^1*. Furthermore, we have:*

the coefficients of $A(\xi)$ *are*

$$a_{is}(\xi) = w(c_i, c_s)(\xi) - w(c_m, c_s)(\xi) - \mathbb{L}_{c_s} \sum_{l=1}^{m} \left[w(c_i, c_l)(\xi) - w(c_m, c_l)(\xi) \right]$$

$$= a_{is}(0) + \alpha \sum_{j=1}^{\infty} \xi^j \int_0^1 \int_{c_i}^{c_m} (j + w - v)^{-(\alpha+1)} \left(L_{c_s}(v) - \mathbb{L}_{c_s} \right) \, dw \, dv \quad (i \neq m \,,\ s \neq m) \,;$$

the coefficients of $\vec{a_2}(\xi)$*are*

$$a_{ms}(\xi) = \sum_{k=1}^{m} \mathbb{L}_{c_k} \left(w\left(c_k, c_s\right)(\xi) - \mathbb{L}_{c_s} \sum_{l=1}^{m} w\left(c_k, c_l\right)(\xi) \right)$$

$$= a_{ms}(0) + \sum_{i=1}^{m-1} \mathbb{L}_{c_i}(a_{is}(\xi) - a_{is}(0)) + \sum_{j=1}^{\infty} \xi^j \int_0^1 (j + c_m - v)^{-\alpha} \left(L_{c_s}(v) - \mathbb{L}_{c_s}\right) dv \quad (s \neq m) \; ;$$

the coefficients of $\vec{a_1}(\xi)$ *are*

$$a_{im}(\xi) = \sum_{l=1}^{m} w\left(c_i, c_l\right)(\xi) - w\left(c_m, c_l\right)(\xi) = -(1-\xi) \int_{c_i}^{c_m} \Phi(\xi, \alpha, w) \, dw \quad (i \neq m)$$

and

$$a_{mm}(\xi) = \sum_{k=1}^{m} \mathbb{L}_{c_k} \sum_{l=1}^{m} w\left(c_k, c_l\right)(\xi) = \sum_{i=1}^{m-1} \mathbb{L}_{c_i} a_{im}(\xi) + r(\xi)$$

with $r(\xi) = \int_0^{c_m} (c_m - v)^{-\alpha} \; dv + \sum_{j=1}^{\infty} \xi^j \int_0^1 (j + c_m - v)^{-\alpha} \; dv = \int_0^{c_m} (c_m - v)^{-\alpha} \; dv + \int_0^1 \xi \Phi(\xi, \alpha, c_m - v + 1) \; dv.$

Only $r(\xi)$ *has no coefficient sequence in* ℓ^1*.*

4. Bounds for the stability domain

We establish estimates for the stability domain depending on α and the collocation parameters, to deduce simple conditions for $A(\pi/2)$–stability and to obtain useful predictions for the stability behavior. For example the estimates can be used to choose a stepsize h for any $\lambda \in \mathbb{C}^-$, so that the collocation method is stable.

Since the first $m-1$ rows of the transformed matrix $BW(\xi)B^{-1}$ have absolutely convergent coefficients for all $|\xi| \leq 1$, $m-1$ eigenvalues can be estimated by Gerschgorin's theorem. The mth eigenvalue is an unbounded function of ξ. However, in analyzing the stability domain we have to consider the reciprocal of the eigenvalues. Let us define $\overline{B_R(y)} := \mathbb{C} \setminus B_{-R}(y)$ for $R < 0$ and $\overline{B_\infty(\infty)} := \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\}$ for $R = \infty$. Then we can state that the function $f : z \mapsto \frac{1}{z}$ is a bijective map from $\left\{z \in \mathbb{C} \cup \{\infty\} \mid \text{ Re } z \geq \frac{1}{2R}\right\}$ into $\overline{B_R(R)}$ for every $R \in \mathbb{R} \cup \{\infty\}$. Hence we observe $\mathbb{C} \setminus \left\{\frac{1}{z} \in \mathbb{C} \mid \text{ Re } z \geq \frac{1}{2R}\right\} = B_{-R}(R)$.

Corollary 4.1. *There exist R_m^- and R_i , R_i^+ , R_i^- for $i = 1, \ldots, m-1$, all different from zero, such that*

$$S \supseteq B_{-R_m^-}(R_m^-) \cap \bigcap_{i=1}^{m-1} \left(B_{R_i}(0) \cup B_{-R_i^-}(R_i^-) \cup B_{R_i^+}(R_i^+)\right) \quad .$$

The stability domain is not empty and includes at least a disk $B_r(-r)$ with $r > 0$. If all R_i^- are positive, the collocation method is $A(\pi/2)-$stable, i.e., S includes $\mathbb{C}^-$.

Proof. Using the theorem of Gerschgorin each eigenvalue z of any matrix A satisfies the inequality $\sum\limits_{\substack{k=1\\k\neq j}}^{m} |a_{jk}| \geq |z - a_{jj}| \geq |\text{ Re } (z - a_{jj})|$ for some $j \in \{1, \ldots, m\}$. For the eigenvalues z of $BW(\xi)\, B^{-1}$ this leads to

$$\frac{1}{2R_i^-} \leq \text{ Re } a_{ii}(\xi) - \sum_{\substack{k=1\\k\neq i}}^{m} |a_{ik}(\xi)| \leq \text{ Re } z \leq \text{ Re } a_{ii}(\xi) + \sum_{\substack{k=1\\k\neq i}}^{m} |a_{ik}(\xi)| \leq \frac{1}{2R_i^+}$$

$$\text{and} \qquad |z| \leq \sum_{k=1}^{m} |a_{ik}(\xi)| \leq \frac{1}{R_i} \qquad \text{for one } i \in \{1, \ldots, m-1\} \quad ,$$

$$\text{or} \qquad \text{Re } z \geq \text{ Re } a_{mm}(\xi) - \sum_{k=1}^{m-1} |a_{mk}(\xi)| \geq \frac{1}{2R_m^-} \quad .$$

Except for $a_{mm}(\xi)$, all $a_{ik}(\xi)$ have coefficient sequences in ℓ^1. Thus, there exist R_i^-, R_i^+, and R_i different from zero. $R_m^- \neq 0$ is a consequence of the given estimates in the section below. Considering again the reciprocal of the eigenvalues, this shows

$$S \supseteq \mathbb{C} \setminus \left(\overline{B_{R_m^-}(R_m^-)} \cup \bigcup_{i=1}^{m-1} \left(\overline{B_{-R_i}(0)} \cap \left(\overline{B_{R_i^-}(R_i^-)} \setminus B_{R_i^+}(R_i^+)\right)\right)\right)$$

$$= B_{-R_m^-}(R_m^-) \cap \bigcap_{i=1}^{m-1} \left(B_{R_i}(0) \cup B_{-R_i^-}(R_i^-) \cup B_{R_i^+}(R_i^+)\right) \quad .$$

The sufficiency for $A(\pi/2)-$stability is an immediate consequence of $S \supseteq \mathbb{C} \setminus \bigcup\limits_{i=1}^{m} \overline{B_{R_i^-}(R_i^-)}$. ∎

While this corollary yields the existence of estimates, in the following discussion we give bounds for the coefficients of $BW(\xi)\,B^{-1}$, and therefore we provide possible choices for R_i, R_i^{+}, R_i^{-}. For easier reading we assume $i \neq m$ and $s \neq m$.

The estimates of $a_{im}(\xi)$ and $a_{mm}(\xi)$ reduce to the estimates of $-(1-\xi)\,\Phi(\xi,\alpha,w)$ and $\xi\Phi(\xi,\alpha,w)$, while $a_{is}(\xi)$ and $a_{ms}(\xi)$ need the estimate of $\int_0^1 (L_{c_s}(v) - I\!L_{c_s})\,\xi\Phi(\xi,r,w-v+1)\,dv$. The problem with good estimates of the last term is the lack of information about the sign changes of $L_{c_s}(v) - I\!L_{c_s}$.

The integral representation of Φ (Erdélyi [1953]) and results for convex series (Nevanlinna [1976]), which can be derived from special results of Fourier transformation (Katznelson [1968]), yield

$$\frac{1}{2}(1+x)^{-r} - x^{-r} \;\leq\; -\Phi(-1,r,x) \;\leq\; \mathrm{Re}\,\xi\;\Phi(\xi,r,x)$$

and

$$0 \;\leq\; \mathrm{Re}\,(1-\xi)\;\Phi(\xi,r,x) \;\leq\; |(1-\xi)\;\Phi(\xi,r,x)| \;\leq\; 2\cdot\Phi(-1,r,x) \;\leq\; 2x^{-r}-(1+x)^{-r}\;.$$

Hence we obtain

$$|a_{im}(\xi)| \leq -a_{im}(-1) = 2\int_{c_i}^{c_m} \Phi(-1,\alpha,w)\,dw \tag{4.1}$$

$$\leq \int_{c_i}^{c_m} 2w^{-\alpha} - (1+w)^{-\alpha}\,dw = \frac{1}{1-\alpha}\left(2\left(c_m^{1-\alpha}-c_i^{1-\alpha}\right) - \left((1+c_m)^{1-\alpha}-(1+c_i)^{1-\alpha}\right)\right)\;,$$

$$\text{and} \quad a_{im}(-1) \leq \;\mathrm{Re}\,a_{im}(\xi) \;\leq\; a_{im}(1) = 0\;. \tag{4.2}$$

Furthermore,

$$\mathrm{Re}\,a_{mm}(\xi) \geq r(-1) + \frac{1}{2}\sum_{i=1}^{m-1}(I\!L_{c_i} + |I\!L_{c_i}|)\cdot a_{im}(-1)$$

$$= \frac{1}{2}\sum_{i=1}^{m-1}(I\!L_{c_i} + |I\!L_{c_i}|)\cdot a_{im}(-1) + \int_0^{c_m}(c_m-v)^{-\alpha}\;dv - \int_0^1 \Phi(-1,\alpha,c_m-v+1)\,dv \tag{4.3}$$

$$\geq \frac{1}{2}\sum_{i=1}^{m-1}(I\!L_{c_i} + |I\!L_{c_i}|)\cdot a_{im}(-1) + \frac{1}{1-\alpha}\cdot\left(2\cdot c_m^{1-\alpha} - \frac{3}{2}\cdot(1+c_m)^{1-\alpha} + \frac{1}{2}\cdot(2+c_m)^{1-\alpha}\right)\;.$$

To get estimates for $a_{is}(\xi)$ and $a_{ms}(\xi)$ we consider $\int_0^1 (L_{c_s}(v) - I\!L_{c_s})(j+w-v)^{-r}\ dv$. We obtain

$$\int_0^1 (L_{c_s}(v) - I\!L_{c_s})(j+w-v)^{-r}\,dv \le \begin{cases} r\cdot l_s \int_0^1 (j+w-v)^{-r-1}\ dv = l_s((j+w-1)^{-r} - (j+w)^{-r}) \\ \qquad\qquad \text{for } j+w>1\ , \\ l_s \int_0^1 (j+w-v)^{-r}\ dv = l_s \frac{1}{1-r}((j+w)^{1-r} - (j+w-1)^{1-r}), \end{cases}$$

with $l_s := \max_{v\in[0,1]} |L_{c_s}(v) - I\!L_{c_s}| \ \ge\ \left|\int_0^v L_{c_s}(w) - I\!L_{c_s}\ dw\right|$ (in the top case we used the product rule).

Furthermore, since the upper estimate is better than the second for $j+w>1+r$, we split the sums involved in $a_{is}(\xi)$ and $a_{ms}(\xi)$ over j. The following sum is a convergent telescoping sum and we get

$$\left|\sum_{j=3}^{\infty} \xi^j \int_0^1 (L_{c_s}(v) - I\!L_{c_s})(j+w-v)^{-r}\ dv\right| \ \le\ r\cdot l_s \sum_{j=3}^{\infty}\int_0^1 (j+w-v)^{-r-1}\ dv = l_s\,(2+w)^{-r}\ .$$

For $a_{ms}(\xi)$ we have $w=c_m>0$ and $r=\alpha$, which implies $2+c_m>1+\alpha$ for all $\alpha\in[0,1)$. Hence, we use the upper estimate for $\int_0^1 (L_{c_s}(v) - I\!L_{c_s})(2+w-v)^{-r}\ dv$ as well. We obtain

$$|a_{ms}(\xi)| \le |a_{ms}(0)| + \sum_{i=1}^{m-1} |I\!L_{c_i}|\cdot|a_{is}(\xi) - a_{is}(0)| + l_s\cdot\begin{cases} c_m^{-\alpha} \\ (1+c_m)^{-\alpha} + \frac{1}{1-\alpha}((1+c_m)^{1-\alpha} - c_m^{1-\alpha})\ . \end{cases} \tag{4.4}$$

For $a_{is}(\xi) - a_{is}(0)$ we have $w\in(c_i,c_m)$ and $r=\alpha+1$, which implies that we need $j\ge 3$. This yields

$$\begin{aligned}
&|a_{is}(\xi) - a_{is}(0)| \\
&\le l_s\cdot\alpha\left[\int_{c_i}^{c_m} -\frac{1}{\alpha}\left((1+w)^{-\alpha} - w^{-\alpha}\right) + \left\{\begin{matrix} ((1+w)^{-(\alpha+1)} - \alpha(2+w)^{-(\alpha+1)} \\ -\frac{1}{\alpha}((2+w)^{-\alpha} - (1+w)^{-\alpha} \end{matrix}\right\} + (2+w)^{-(\alpha+1)}\ dw\right] \\
&= l_s\cdot\left(\frac{1}{1-\alpha}\cdot(c_m^{1-\alpha} - c_i^{1-\alpha})\right. \qquad (4.5) \\
&\quad \left. -\frac{1}{1-\alpha}\left((q+c_m)^{1-\alpha} - (q+c_i)^{1-\alpha}\right) + \left((q+c_i)^{-\alpha} - (q+c_m)^{-\alpha}\right)\right) \quad \text{with } q=1 \text{ or } q=2 \\
&\le l_s\cdot\frac{1}{1-\alpha}\cdot(c_m^{1-\alpha} - c_i^{1-\alpha})\quad .
\end{aligned}$$

Again, the difficulty in estimating $\operatorname{Re}\ a_{ii}(\xi)$ is due to the unknown sign changes of $L_{c_s}(v) - \mathbb{L}_{c_s}$. Therefore, we can only derive

$$|a_{ii}(\xi)| \le |a_{ii}(0)| + |a_{ii}(\xi) - a_{ii}(0)|$$

and

$$a_{ii}(0) - |a_{ii}(\xi) - a_{ii}(0)| \ \le\ \operatorname{Re}\ a_{ii}(\xi)\ \le\ a_{ii}(0) + |a_{ii}(\xi) - a_{ii}(0)| \quad . \tag{4.6}$$

Summarized (4.1) – (4.6) give the needed estimates for $|a_{is}(\xi)|, |a_{im}(\xi)|, |a_{ms}(\xi)|$, $\operatorname{Re}\ a_{ii}(\xi)$ and $\operatorname{Re}\ a_{mm}(\xi)$.

5. Applications and numerical examples

For collocation approximation with piecewise constants, with piecewise linear functions, and with continuous splines, we analyze the estimates in detail and note the conclusions for the stability behavior. Furthermore, we exploit Theorem 2.4 to plot the boundary of the stability domain and compare it with the developed estimates. In the figures the boundary of the stability domain is evaluated numerically by the reciprocal of the eigenvalues of the series $\sum_{j=0}^{N} BW_jB^{-1}\xi^j$ for ξ with $|\xi| = 1$, $\operatorname{Im}(\xi) \ge 0$, and equally spaced imaginary parts, using fixed N. These inverses of the eigenvalues are interpolated by cubic splines for $\xi = e^{i\varphi}$ on the unit circle from $\varphi = 0$ to $\varphi = 2\pi$, such that the curve of the boundary is smooth except at $\xi = 1$, where it is at least continuous. The marked points are the computed values.

5.1. Collocation approximation with splines of degree zero: If we use the collocation approximation of Volterra integral equation (1.1) with splines of degree 0, the numerical solution u is piecewise constant and is defined in $[t_n, t_{n+1})$ by $u(t) = u(t_n + c_1h)$ with

$$u(t_n + c_1h) = g(t_n + c_1h) + \lambda h^{1-\alpha}\sum_{j=0}^{n} w_{n-j}(c_1)u(t_j + c_1h) ,$$

where $w_0(c_1) = \int_0^{c_1}(c_1 - v)^{-\alpha}\, dv$ and $w_j(c_1) = \int_0^{1}(j + c_1 - v)^{-\alpha}\, dv$ for $j \ge 1$. We can state

Corollary 5.1.1. *The stability domain S is characterized by $S = \mathbb{C} \setminus \left\{ \frac{1}{w(\xi)} \text{ with } |\xi| \leq 1 \right\}$ with*

$w(\xi) = \frac{1}{1-\alpha} \left(c_1^{1-\alpha} + \sum_{j=1}^{\infty} \left((j+c_1)^{1-\alpha} - (j+c_1-1)^{1-\alpha} \right) \cdot \xi^j \right)$. *Furthermore, for all $\xi \in \mathbb{C}$ with $|\xi| \leq 1$ we have*

$$\operatorname{Re} w(\xi) \geq w(-1) \geq \frac{1}{1-\alpha} \left(2 \cdot c_1^{1-\alpha} - \frac{3}{2} \cdot (1+c_1)^{1-\alpha} + \frac{1}{2} \cdot (2+c_1)^{1-\alpha} \right) .$$

Analyzing the function $g(c_1, \alpha) := \frac{1}{1-\alpha} \left(2 \cdot c_1^{1-\alpha} - \frac{3}{2} \cdot (1+c_1)^{1-\alpha} + \frac{1}{2} \cdot (2+c_1)^{1-\alpha} \right)$ we obtain: *For all $\alpha \in [0,1)$ and $c_1 \in [0,1]$ we have $w(-1) \geq \frac{1}{1-\alpha} \left(2^{-\alpha} - \frac{3}{2} \right)$ and $w(-1) \geq \frac{1}{1-\alpha} (c_1 - 1)$. If $c_1 \geq 0.2928$ then it follows that $w(-1) \geq c_1 - \frac{1}{2}$. For $1 \geq c_1 \geq \frac{3}{2} \cdot (1-\alpha)$ $\quad (\Rightarrow \alpha \geq \frac{1}{3})$ we obtain $w(-1) \geq 0$. $c_1 = 1$ leads to $w(-1) \geq \frac{1}{1-\alpha} \cdot \frac{1}{2}$. For all $c_1 \neq 0$ we have $\lim_{\alpha \to 1} w(-1) = +\infty$, while $c_1 = 0$ leads to $w(-1) \leq -\frac{1}{1-\alpha} \cdot \frac{1}{2}$ and $\lim_{\alpha \to 1} w(-1) = -\infty$.* An immediate consequence of these estimates is

Corollary 5.1.2. *The stability domain S includes at least $\mathbb{C} \setminus \overline{B_R(R)}$ where $R = \frac{1}{2w(-1)}$ and this is the maximal stability disk. For $c_1 \geq \frac{1}{2}$ and $1 \geq c_1 \geq \frac{3}{2}(1-\alpha)$ the method is $A\left(\frac{\pi}{2}\right)$–stable, i.e., $S \supseteq \mathbb{C}^-$. If $c_1 = 1$ we have at least $S \supseteq \mathbb{C} \setminus \overline{B_1(1)}$. If $c_1 \neq 0$, then the stability domain converges to $\mathbb{C} \setminus \{0\}$ as $\alpha \to 1$, whereas for $c_1 = 0$ the stability domain converges to the empty set as $\alpha \to 1$.*

Brunner et al. [1991] consider c–methods and Bakke and Jackiewicz [1986] investigate the product θ–method, which —when applied to the test equation— are equivalent to the one–step collocation method. They choose as a test equation the Abel integral equation (1.1) with constant $g(t) \equiv 1, \lambda \in \mathbb{R}$ and $\lambda < 0$ rather than —as in this paper— $\lim_{t \to \infty} g(t) = c$ and $\lambda \in \mathbb{C}$. Although their stability concept considers bounded solutions (while we analyze the asymptotic stability behavior $\lim_{t \to \infty} u(t) = 0$), the above stability results remain valid for their stability concepts, due to the result for bounded solutions in Blank [1993].

We turn to the numerical analysis. In the graphs 1.a–1.f, the stability domain of the theoretical solution (the lines) as well as the stability domain of the method are plotted. They indicate that for $c_1 = 1/2$ the method is A–stable, not only for $\alpha = 0$, but for all α. If we choose $c_1 = 1$ the numerical results are similar to those of $c_1 = 1/2$, though the stability domains are larger.

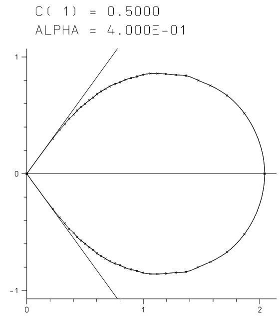

Figure 1.a

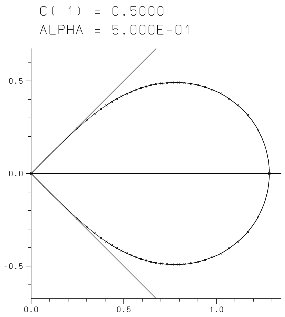

Figure 1.b

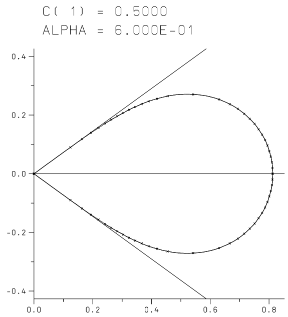

Figure 1.c

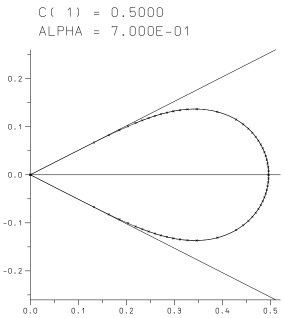

Figure 1.d

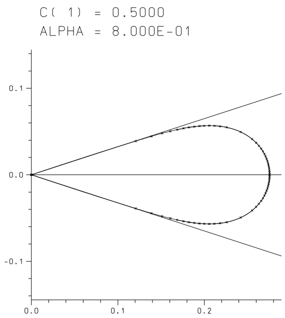

Figure 1.e

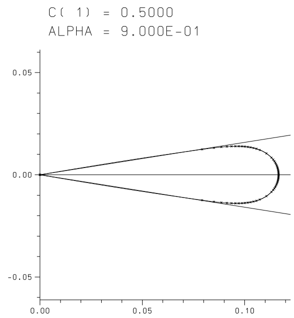

Figure 1.f

The following figures illustrate the variation of the stability domains with fixed $\alpha = 0.5$ and varying collocation parameter c_1. For increasing parameter c_1 the curves intersect first the negative axis (if they do at all) in decreasing real values and then the positive axis in decreasing real values. In figure 2.a the stability boundaries are plotted for $c_1 = 0.0, \ldots, 0.4$ and in figure 2.b for $c_1 = 0.4, \ldots, 1.0$. It seems that the stability domain behaves monotonically with respect to c_1 for $\alpha = 0.5$, i.e., the stability domain for any c_1 includes the stability domains for smaller c_1.

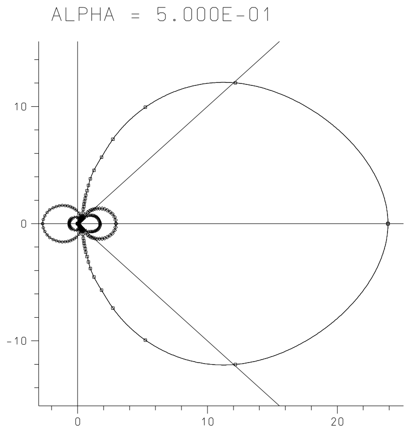

Figure 2.a

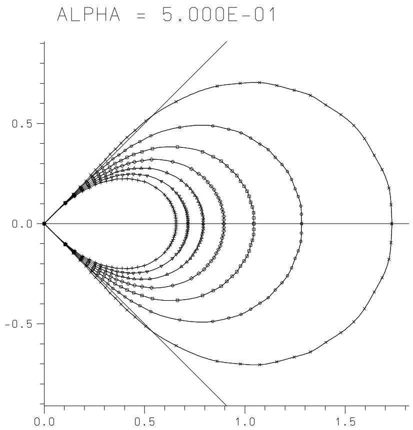

Figure 2.b

In figures 3.a–3.d we compare the boundary of the stability domain with the estimates. The plotted circles $B_r(r)$ are determined by the given bounds of $w(-1)$. Except for the case where the circle lies in the negative complex plane and the boundary of the stability domain is contained in $\mathbb{C}^+$ the estimates are good.

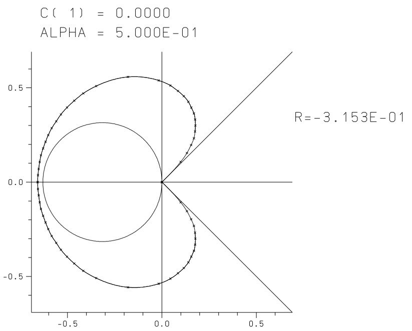

Figure 3.a

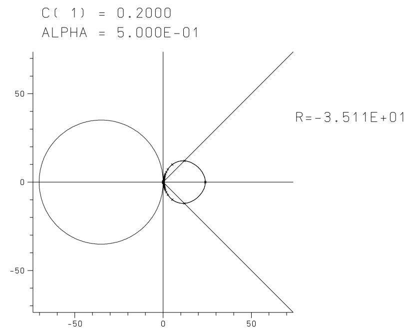

Figure 3.b

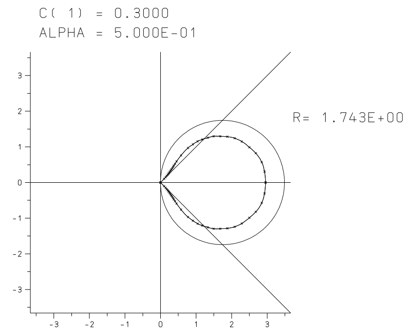

Figure 3.c

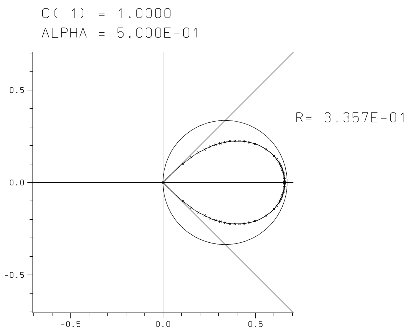

Figure 3.d

5.2. Collocation approximation with splines of degree one: The stability domain of the collocation approximation with splines of degree one ($u \in S_2(Z)$) is characterized by the reciprocals of the zeros of the normed polynomial $z^2 - (a_{11}(\xi) + a_{22}(\xi)) \cdot z + (a_{11}(\xi) \cdot a_{22}(\xi) - a_{12}(\xi) \cdot a_{21}(\xi))$. For polynomials of degree two we can derive a different estimate for the real values of the zeros:

$$-\sqrt{|a_{12}(\xi)a_{21}(\xi)|} + \text{ Re } a_{11}(\xi) \leq \text{ Re } z \leq \text{ Re } a_{11}(\xi) + \sqrt{|a_{12}(\xi)a_{21}(\xi)|}$$

$$\text{or} \qquad -\sqrt{|a_{12}(\xi)a_{21}(\xi)|} + \text{ Re } a_{22}(\xi) \leq \text{ Re } z \quad .$$

These will be combined with the estimates of the previous section, which were based on Gerschgorin's Theorem. Furthermore, $\mathrm{L}_{c_1}(v) - \mathbb{L}_{c_1} = \frac{1}{c_1 - c_2}\left(v - \frac{1}{2}\right)$ is symmetric on $[0,1]$ with respect to $x = \frac{1}{2}$, which is used for the calculation of l_1. Plotted examples for the numerical comparison of these estimates with the boundaries of the stability domains are given below. Only the relevant circles are graphed. The shaded regions represent the domains which belong at least to the stability domain of the method on account of the estimates. Particularly in figure 4.g we see that two circles represent the estimates of the reciprocals of two different functions of eigenvalues.

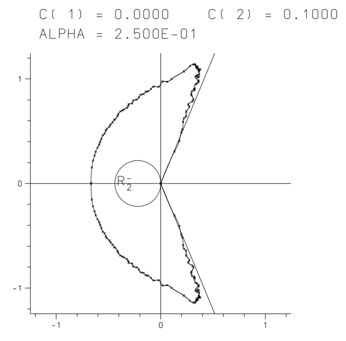

Figure 4.a

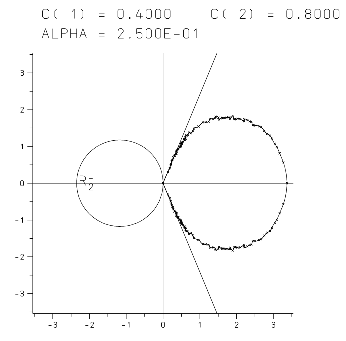

Figure 4.b

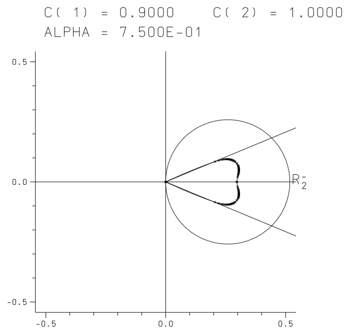

Figure 4.c

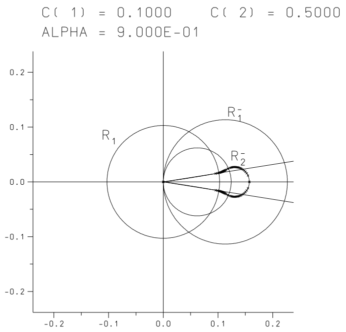

Figure 4.d

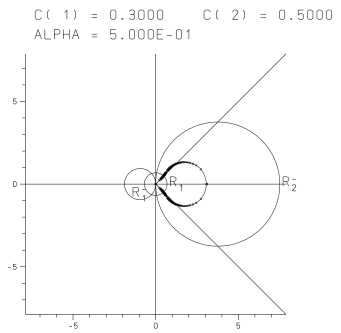

Figure 4.e

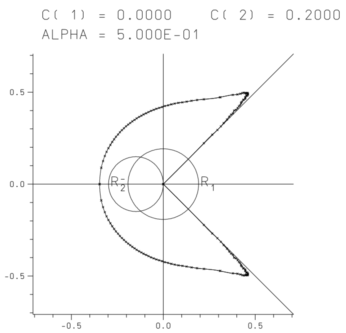

Figure 4.f

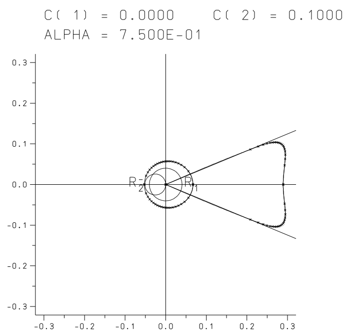

Figure 4.g

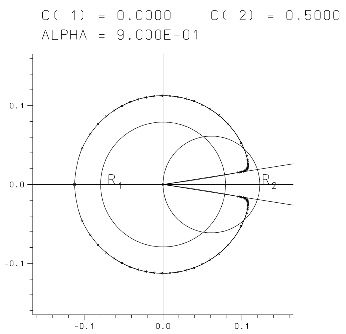

Figure 4.h

5.3. Collocation approximation with continuous splines: If we choose the collocation solution $u \in S_m(Z)$ with $c_1 = 0$ and $c_m = 1$, then u is a continuous function. Moreover, we can reduce the $m-$dimensional collocation equation to an $(m-1)-$dimensional equation in the following way:

We have $w_j(c_1, c_l) = w_{j-1}(c_m, c_l)$ for $j \geq 1$, which leads to

$$w(c_1, c_l)(\xi) = \sum_{j=1}^{\infty} w_j(c_1, c_l)\xi^j = \sum_{j=1}^{\infty} w_{j-1}(c_m, c_l)\xi^j = \xi \cdot w(c_m, c_l)(\xi) \ .$$

Let $u_l(\xi)$ denote the lth component of $\vec{u}(\xi)$ then, as a result of the continuity, we can rewrite

$$u_1(\xi) = \sum_{j=0}^{\infty} u(t_j + c_1 h)\xi^j = g(0) + \sum_{j=1}^{\infty} u(t_{j-1} + c_m h)\xi^j = g(0) + \xi \cdot u_m(\xi) \ .$$

For $i = 2, \ldots, m$ we obtain:

$$u_i(\xi) = \tilde{g}_i(\xi) + \lambda h^{1-\alpha} \sum_{l=2}^{m} \tilde{w}(c_i, c_l)(\xi) u_l(\xi) \ ,$$

where $\tilde{w}(c_i, c_l)(\xi) := w\,(c_i, c_l)\,(\xi)$ for $l = 2, \ldots, m-1$, $\tilde{w}(c_i, c_m)(\xi) := \xi \cdot w\,(c_i, c_1)\,(\xi) + w\,(c_i, c_m)\,(\xi)$, and $\tilde{g}_i(\xi) := g_i(\xi) + \lambda h^{1-\alpha} g(0) w\,(c_i, c_1)\,(\xi)$.

Since $u(t_n + c_1 h) = u(t_{n-1} + c_m h)$, it is not neccessary to compute $u(t_n + c_1 h)$. The collocation approximation u is determined by an $(m-1)-$dimensional system of linear equations and we define $\tilde{W}(\xi) \in \mathbb{C}^{(m-1)\times(m-1)}$ as $\tilde{W}(\xi) = (\tilde{w}(c_i, c_l)(\xi))_{\substack{i=2,\ldots,m \\ l=2,\ldots,m}}$. The eigenvalues of $W(\xi)$ are 0 or eigenvalues of $\tilde{W}(\xi)$, hence we have a characterization of the stability domain, which depends on an $(m-1) \times (m-1)$ dimensional matrix.

Theorem 5.3.1. *The stability domain S of the collocation approximation with continuous splines is given by*

$$S = \mathbb{C} \setminus \left\{ z \in \mathbb{C} \,\middle|\, \frac{1}{z} \ \textit{is an eigenvalue of} \ \tilde{W}(\xi) \ \textit{for} \ \xi \in \mathbb{C} \ \textit{with} \ |\xi| \leq 1 \right\} \ .$$

In the simplest case we choose $m = 2$, i.e. $c_1 = 0$ and $c_2 = 1$ which are as well the Lobatto points on $[0,1]$. Then the spline is a piecewise linear function interpolating $u_n := u(t_n)$ and is determined by

$$u_0 = g(0) \quad \text{and} \quad u_n = \tilde{g}(t_n) + \lambda h^{1-\alpha} \sum_{j=0}^{n} r_{n-j} u_j \quad \text{for } n \geq 1$$

with $\tilde{g}(t_n) := g(t_n) + \lambda h^{1-\alpha} g(0) \cdot w_n(1,0)$ and $r(\xi) := \frac{1}{1-\alpha} \cdot \frac{1}{2-\alpha} \{1 + \sum_{j=1}^{\infty} \left((j+1)^{2-\alpha} - 2j^{2-\alpha} + (j-1)^{2-\alpha}\right)\}$

for $j \geq 1$. We obtain $\operatorname{Re} r(\xi) \geq r(-1) \geq \frac{1}{1-\alpha} \cdot \frac{1}{2-\alpha} \cdot \frac{1}{2} \cdot \left(7 + 3^{2-\alpha} - 4 \cdot 2^{2-\alpha}\right) \geq 0$ and $\lim_{\alpha \to 1} r(-1) = +\infty$.

Therefore, we have results corresponding to the case $m = 1$. Furthermore, we obtain:

Corollary 5.3.2. *The collocation method with $c_1 = 0$ and $c_2 = 1$ is $A\left(\frac{\pi}{2}\right)-$stable and the stability domain converges to $\mathbb{C} \setminus \{0\}$ as $\alpha \to 1$.*

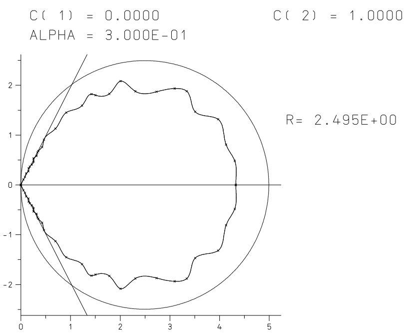

Figure 5.a

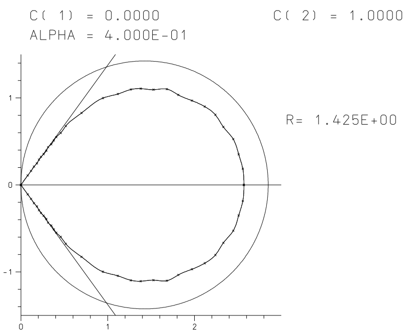

Figure 5.b

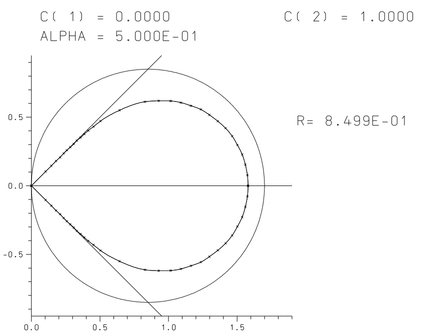

Figure 5.c

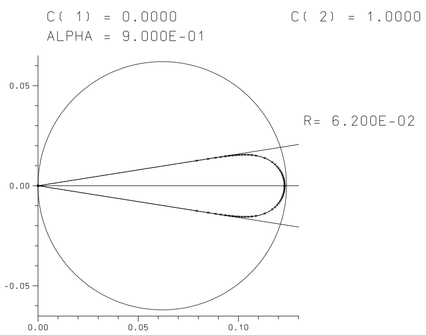

Figure 5.d

In the above figures, the boundaries of the stability domains of the Abel integral equation (the lines) and the stability domains of the numerical method are plotted. The estimates are illustrated by circles. We note that for small α the evaluation of the series is inaccurate but the estimates give good suggestions for the choice of h for stable $\lambda h^{1-\alpha}$. We can see the convergence of the stability domain to $\mathbb{C} \setminus \{0\}$ as $\alpha \to 1$ and observe that the estimates for the maximal stability disks are good. Moreover, we conjecture that the method employing the Lobatto points $c_1 = 0$ and $c_2 = 1$ is $A-$stable for all $\alpha \in [0,1)$.

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